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Global solutions of the Klein–Gordon–Schrödinger system with rough data in $\mathbb{R}^{2+1}$ ☆

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Abstract

In this paper, we consider the Klein–Gordon–Schrödinger system with quadratic (Yakuwa) coupling and cubic autointeractions in $\mathbb{R}^{2+1}$, and prove the existence and uniqueness of global solution for rough data. The techniques to be used are adapted from a general scheme originally introduced by J. Bourgain to split the data into the low and high frequency parts.

Keywords: Klein–Gordon–Schrödinger system; Strichartz-type estimates; Besov space; Global solution; Cauchy problem

1. Introduction

Let us consider the Cauchy problem for the Klein–Gordon–Schrödinger system

$$\begin{cases} i\psi_t + \Delta\psi = -\phi\psi + |\psi|^2\psi, \\ \phi_{tt} + (-\Delta + 1)\phi = |\psi|^2 - \phi^3, \\ \psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \phi_t(0) = \phi_1. \end{cases} \tag{1.1}$$

Here $\psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ is the nucleon field and $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the meson field.

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The system (1.1) without cubic autointeractions have been solved in three dimensions [3,12–14]. H. Pecher studied the global solution without finite energy for 1-dimensional Zakharov system by Fourier Truncation Method in [24] and recently by I-method in [26]. At the same time, H. Pecher considered the same case for 3-dimensional system (1.1) without cubic autointeractions in [25]. As in [2], we consider the system with quadratic coupling and cubic autointeractions.

Similar in [1], the local well-posedness and energy method imply the existence of a unique global solution for data $\psi_0 \in H^1(\mathbb{R}^2)$, $\phi_0 \in H^1(\mathbb{R}^2)$, $\phi_1 \in L^2(\mathbb{R}^2)$ with $\psi \in C^0(\mathbb{R}, H^1(\mathbb{R}^2))$, $\phi \in C^0(\mathbb{R}, H^1(\mathbb{R}^2)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^2))$ by the following conservation laws:

$$\begin{cases} \|\psi(t)\| =: M(\psi,t) =: M, \\ \|\nabla\psi(t)\|^2 + \frac{1}{2}(\|A\phi(t)\|^2 + \|\phi_t(t)\|^2) - \int_{\mathbb{R}^2} |\psi(t)|^2\phi(t)\,dx \\ \quad + \frac{1}{2}\int_{\mathbb{R}^2} |\psi|^4\,dx + \frac{1}{4}\int_{\mathbb{R}^2} |\phi|^4\,dx =: E(\psi,\phi,\phi_t,t) =: E, \end{cases} \tag{1.2}$$

where $\|\cdot\|$ denotes the norm of $L^2(\mathbb{R}^2)$ and A denotes $(I-\Delta)^{\frac{1}{2}}$.

We use Bourgain's idea [5,7] to split the data into the low and high frequency parts in order to prove the global well-posedness for rough data for which the energy method is not directly applicable. This technique has been successfully applied to other problems [8–11, 18,19,22,24].

We also rely on [15] for the local well-posedness (cf. Theorem 3.1) and the framework for the technique. The key point is a similar smoothing estimate as in [25] for the nonlinearity $\psi\phi$ in system (1.1) given in Lemma 2.6 which is a generalized bilinear Strichartz-type estimate. A similar estimate has been given by Bourgain for the Schrödinger equation [6, 7]. We also refer to [17].

Here we introduce the following notation. For $\lambda \in \mathbb{R}^n$, $\langle\lambda\rangle$ denotes $(1+|\lambda|^2)^{\frac{1}{2}}$; $a+$ (respectively $a-$) denotes a number slightly larger (respectively smaller) than a; $\widehat{\ } = \mathcal{F}$ denotes the time–space Fourier transformation without special statement; $\mathcal{F}_t$ denotes the time Fourier transformation; $\mathcal{F}_x$ denotes the space Fourier transformation; $A = (I-\Delta)^{\frac{1}{2}}$, $U(t) = e^{it\Delta}$, $U_\pm(t) = e^{\pm itA}$.

The paper is organized as follows.

In Section 2, we give the bilinear estimates along with Ginibre et al. [15] in the $X^{s,b}$ spaces, which were introduced by Bourgain [5], Klainerman and Machedon [20,21], Kenig et al. [18] and Grünrock [17].

For the free dispersive equation of the form

$$iu_t - \varphi(D_x)u = 0, \quad D_x = -i\nabla_x, \tag{1.3}$$

where φ is a measurable function, let $X^{s,b}_\varphi$ be the completion of $S(\mathbb{R}^3)$ with respect to

$$\|f\|_{X^{s,b}_\varphi} := \big\|\langle\xi\rangle^s\langle\tau\rangle^b\mathcal{F}\big(e^{it\varphi(D_x)}f(x,t)\big)\big\|_{L^2_{\xi,\tau}} = \big\|\langle\xi\rangle^s\big\langle\tau+\varphi(\xi)\big\rangle^b\hat{f}(\xi,\tau)\big\|_{L^2_{\xi,\tau}}.$$

In general, we use the notation $X^{s,b}_\pm$ for $\varphi(\xi) = \pm\langle\xi\rangle$ and $X^{s,b}$ for $\varphi(\xi) = |\xi|^2$ without confusion. For a given time interval I we define:

$$\|f\|_{X^{s,b}(I)} = \inf_{\tilde{f}|_I = f} \|\tilde{f}\|_{X^{s,b}}, \quad \text{where } \tilde{f} \in X^{s,b};$$

$$\|f\|_{X_{\pm}^{s,b}(I)} = \inf_{\tilde{f}|_I = f} \|\tilde{f}\|_{X_{\pm}^{s,b}}, \quad \text{where } \tilde{f} \in X_{\pm}^{s,b}.$$

In Section 3, we transform the system into a first order system for (ψ_0, ϕ_+, ϕ_-) and give the local well-posedness result by virtue of the results of [15] for the Zakharov system.

In Section 4, we split the data into sums $\psi_0 = \psi_{01} + \psi_{02}$, $\phi_0 = \phi_{01} + \phi_{02}$, $\phi_1 = \phi_{11} + \phi_{12}$, where the low frequency parts $(\psi_{01}, \phi_{01}, \phi_{11})$ are regular with large norms and the high frequency parts $(\psi_{02}, \phi_{02}, \phi_{12})$ are just in $H^s(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2)$ with small L^2-norms.

In Section 5, the solution $(\tilde{\psi}, \tilde{\phi}_+, \tilde{\phi}_-)$ of the corresponding first order system with data $(\psi_{01}, \phi_{01}, \phi_{11})$ is further investigated on a suitable time interval I using the conservation laws and Strichartz-type estimates, we get further bounds about them.

In Section 6, we consider the system fulfilled by $(\hat{\psi}, \hat{\phi}_\pm) = (\psi - \tilde{\psi}, \phi_\pm - \tilde{\phi}_\pm)$ with data $(\psi_{02}, \phi_{0\pm 2})$ and construct a solution on the same interval I, thus giving a solution of our original system on I. Noting that the resolution space in 2-dimensional case is more natural than that in 3-dimensional case [25]. In addition, the inhomogeneous parts w of $\hat{\psi}$ and $z_\pm$ of $\hat{\phi}_\pm$ are shown to belong to $H^1(\mathbb{R}^2)$, thus to be more regular than the homogeneous parts $e^{it\Delta}\psi_{02}$ and $e^{\mp itA}\phi_{0\pm 2}$ which belong to $H^s(\mathbb{R}^2)$ and $H^m(\mathbb{R}^2)$, respectively. This shows that Bourgain's argument is applicable, which relies on the nonlinearity part of the equation to be smoother than the linear part.

In Section 7, we show that the process can be iterated to get a solution on any time interval $[0, T]$. We construct solutions step by step on time intervals of equal length $|I|$, taking as new initial data at time $|I|$ the triple $(\tilde{\psi}(|I|) + w(|I|), \tilde{\phi}_\pm(|I|) + z_\pm(|I|))$ and repeating the argument on $[|I|, 2|I|]$. In each step the involved norms have to be controlled independently of the iteration step in order to be able to choose intervals of equal length.

We use the following standard facts about the spaces $X_\varphi^{s,b}$.

First we have for any $s \in \mathbb{R}$,

$$\|f\|_{X_\varphi^{s,0}(I)} \leqslant \|f\|_{L_t^\infty(I, H_x^s)} |I|^{\frac{1}{2}} \leqslant c\|f\|_{X_\varphi^{s,(1/2)+}(I)} |I|^{\frac{1}{2}} \tag{1.4}$$

by Hölder and Sobolev inequalities.

Next let χ be a nonnegative, radial cut-off function in $C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (-2, 2)$, $\chi \equiv 1$ on $[-1, 1]$, $\chi_\delta(t) = \chi(\frac{t}{\delta})$, if $0 < \delta \leqslant 1$, we have for $b \geqslant 0$,

$$\left\|\chi_1 e^{-i\varphi(D_x)t} f(x)\right\|_{X_\varphi^{s,b}} \leqslant c\|f\|_{H_x^s}; \tag{1.5}$$

for $b' + 1 \geqslant b \geqslant 0 \geqslant b' > -\frac{1}{2}$,

$$\left\|\chi_\delta \int_0^t e^{-i\varphi(D_x)(t-\tau)} F(\tau)\, d\tau\right\|_{X_\varphi^{s,b}} \leqslant c\delta^{1-b+b'} \|F\|_{X_\varphi^{s,b'}}. \tag{1.6}$$

Last, the following interpolation property holds,

$$X_\varphi^{s,b}(I) = \big(X_\varphi^{s_0,b_0}(I), X_\varphi^{s_1,b_1}(I)\big)_{[\theta]},$$

where $0 \leqslant \theta \leqslant 1$, $s = (1-\theta)s_0 + \theta s_1$, $b = (1-\theta)b_0 + \theta b_1$.

We also need the following Strichartz-type estimates for the Schrödinger equation in $\mathbb{R}^2$. For $0 \leqslant \frac{2}{\gamma} = 2(\frac{1}{2} - \frac{1}{\rho}) < 1$, the following estimates hold

$$\big\|U(t)\psi_0\big\|_{L_t^\gamma(I, L_x^\rho(\mathbb{R}^2))} \leqslant c\|\psi_0\|_{L_x^2(\mathbb{R}^2)} \quad \text{and} \quad \|f\|_{L_t^\gamma(I, L_x^\rho(\mathbb{R}^2))} \leqslant c\|f\|_{X^{0,(1/2)+}(I)}. \tag{1.7}$$

Its dual version is

$$\|f\|_{X^{0,-(1/2)-}(I)} \leqslant c\|f\|_{L_t^{\gamma'}(I, L_x^{\rho'}(\mathbb{R}^2))}, \tag{1.8}$$

where γ', ρ' are dual to γ, ρ. Interpolation with the trivial identity $\|f\|_{X^{0,0}(I)} = \|f\|_{L_t^2(I, L_x^2(\mathbb{R}^2))}$ gives

$$\begin{aligned} &\|f\|_{X^{0,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{\gamma'+}(I, L_x^{\rho'+}(\mathbb{R}^2))}, \quad \rho > 2, \\ &\|f\|_{X^{0,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{1+}(I, L_x^2(\mathbb{R}^2))}, \quad \rho = 2; \end{aligned} \tag{1.9}$$

and also

$$\begin{aligned} &\|f\|_{X^{1,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{\gamma'+}(I, H_x^{1,\rho'+}(\mathbb{R}^2))}, \quad \rho > 2, \\ &\|f\|_{X^{1,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{1+}(I, H_x^1(\mathbb{R}^2))}, \quad \rho = 2. \end{aligned} \tag{1.10}$$

Similarly, for the Klein–Gordon equation, for $0 \leqslant \theta \leqslant 1$, let

$$0 \leqslant \frac{2}{\gamma} \leqslant (1+\theta)\left(\frac{1}{2} - \frac{1}{\rho}\right) < 1, \qquad \mu = (2+\theta)\left(\frac{1}{2} - \frac{1}{\rho}\right) - \frac{1}{\gamma},$$

then we have

$$\begin{aligned} &\big\|U_\pm(t)\phi_0\big\|_{L_t^\gamma(I, L_x^\rho(\mathbb{R}^2))} \leqslant c\|\phi_0\|_{H_x^\mu(\mathbb{R}^2)}, \quad \text{and} \\ &\|f\|_{L_t^\gamma(I, L_x^\rho(\mathbb{R}^2))} \leqslant c\|f\|_{X_\pm^{\mu,(1/2)+}(I)}. \end{aligned} \tag{1.11}$$

See [16,23], its dual version is

$$\begin{aligned} &\|f\|_{X_\pm^{-\mu,-(1/2)-}(I)} \leqslant c\|f\|_{L_t^{\gamma'}(I, L_x^{\rho'}(\mathbb{R}^2))}, \\ &\|f\|_{X_\pm^{0,-(1/2)-}(I)} \leqslant c\|f\|_{L_t^{\gamma'}(I, H_x^{\mu,\rho'}(\mathbb{R}^2))}, \end{aligned} \tag{1.12}$$

where γ', ρ' are dual to γ, ρ. Interpolating with the trivial identity $\|f\|_{X_\pm^{0,0}(I)} = \|f\|_{L_t^2(I,L_x^2)}$ gives

$$\begin{aligned} &\|f\|_{X_\pm^{0,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{\gamma'+}(I,H_x^{\mu-,\rho'+}(\mathbb{R}^2))}, && \rho > 2, \\ &\|f\|_{X_\pm^{0,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{1+}(I,L_x^2(\mathbb{R}^2))}, && \rho = 2; \end{aligned} \tag{1.13}$$

and also

$$\begin{aligned} &\|f\|_{X_\pm^{m-1,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{\gamma'+}(I,H_x^{m-1+\mu-,\rho'+}(\mathbb{R}^2))}, && \rho > 2, \\ &\|f\|_{X_\pm^{m-1,-(1/2)+}(I)} \leqslant c\|f\|_{L_t^{1+}(I,H_x^{m-1}(\mathbb{R}^2))}, && \rho = 2. \end{aligned} \tag{1.14}$$

2. Bilinear estimates

In this section we give some bilinear estimates on the linear Schrödinger operator and some coupling bilinear estimates on the linear Schrödinger and Klein–Gordon operators.

Lemma 2.1. *Assume $P \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\nabla P \neq 0$ on $\operatorname{supp}\psi$. Then the following identity holds*:

$$\int_{\mathbb{R}^n} \delta\big(P(x)\big)\psi(x)\,dx = \int_{P(x)=0} \frac{\psi(x)}{|\nabla P(x)|}\,dS_x.$$

Proof. See [25, Lemma 1.1]. □

Lemma 2.2. *For $s \geqslant \frac{1}{2}$, then the following estimates hold*:

$$\begin{aligned} &\big\|U(t)u_1U(t)u_2\big\|_{L_t^2(\mathbb{R},H_x^s(\mathbb{R}^2)} \\ &\quad\leqslant c\|u_1\|_{H_x^s(\mathbb{R}^2)}\|u_2\|_{L_x^2(\mathbb{R}^2)} + c\|u_1\|_{H_x^{(1/2)+}(\mathbb{R}^2)}\|u_2\|_{H_x^{s-1/2}(\mathbb{R}^2)}, \\ &\|u_1u_2\|_{L_t^2(\mathbb{R},H_x^s(\mathbb{R}^2))} \leqslant c\|u_1\|_{X^{s,(1/2)+}}\|u_2\|_{X^{0,(1/2)+}} + c\|u_1\|_{X^{(1/2)+,(1/2)+}}\|u_2\|_{X^{s-1/2,(1/2)+}}. \end{aligned}$$

For $0 < s < \frac{1}{2}$, then the following estimates hold:

$$\begin{aligned} &\big\|U(t)u_1U(t)u_2\big\|_{L_t^2(\mathbb{R},H_x^s(\mathbb{R}^2))} \leqslant c\|u_1\|_{H_x^{s+}(\mathbb{R}^2)}\|u_2\|_{L_x^2(\mathbb{R}^2)}, \\ &\|u_1u_2\|_{L_t^2(\mathbb{R},H_x^s(\mathbb{R}^2))} \leqslant c\|u_1\|_{X^{s+,(1/2)+}}\|u_2\|_{X^{0,(1/2)+}}, \\ &\|u_1u_2\|_{L_t^2(\mathbb{R},H_x^s(\mathbb{R}^2))} \leqslant c\|u_1\|_{X^{s+,(1/2)+}}\|u_2\|_{X^{0,(1/2)-}}, \quad \textit{where } \frac{1}{2}- \textit{ depends on } s+. \end{aligned}$$

Proof. See [6, Corollary 113]. □

Now, we introduce the Littlewood–Paley decomposition: let $\{\phi_j\}_{j=0}^{\infty}$ be smooth resolution of unity in $\mathbb{R}^2$, i.e.

$$\operatorname{supp}\phi_0 \subset \left\{|\xi| \leqslant \frac{5}{6}\right\}, \qquad \operatorname{supp}\phi_j \subset \left\{\frac{3}{5}\,2^{j-1} \leqslant |\xi| \leqslant \frac{5}{3}\,2^{j-1}\right\},$$
$$\sum_{j=0}^{\infty}\phi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,\ 0 \leqslant \phi_j \leqslant 1,\ \phi_j \in C^\infty\big(\mathbb{R}^2\big),\ \forall j, \xi.$$

Define the Littlewood–Paley operators $\Delta_j := \mathcal{F}^{-1}\phi_j\mathcal{F}$ $(j = 0, 1, 2, \ldots)$, $S_{-2} = S_{-1} = 0$, $S_l := \sum_{j=0}^{l}\Delta_j$ $(l = 0, 1, 2, \ldots)$.

Lemma 2.3. *For $l, m \in \mathbf{N} \cup \{0\}$,*

$$\big\|U(t)\Delta_l\psi_1 U_\pm(t)\Delta_m\psi_2\big\|_{L_t^2(\mathbb{R}, L_x^2(\mathbb{R}^2))} \leqslant c2^{\frac{m-l}{2}}\|\Delta_l\psi_1\|_{L_x^2(\mathbb{R}^2)}\|\Delta_m\psi_2\|_{L_x^2(\mathbb{R}^2)}.$$

Proof. Case 1: $l \geqslant 2$. We have

$$\begin{aligned}
&\big\|U(t)\Delta_l\psi_1 U_\pm(t)\Delta_m\psi_2\big\|_{L_{t,x}^2}^2 \\
&\quad = \iint\left|\int e^{-it\{|\xi_1|^2 \mp (|\xi_2|^2+1)^{\frac{1}{2}}\}}\phi_l(\xi_1)\hat{\psi}_1(\xi_1)\ \phi_m(\xi_2)\hat{\psi}_2(\xi_2)\,d\xi_1\right|^2 d\xi\,dt \\
&\quad = \iiiint e^{itP(\eta_1,\xi_1,\xi)}\phi_l(\xi_1)\hat{\psi}_1(\xi_1)\phi_m(\xi_2)\hat{\psi}_2(\xi_2)\phi_l(\eta_1)\bar{\hat{\psi}}_1(\eta_1) \\
&\qquad\qquad \times \phi_m(\eta_2)\bar{\hat{\psi}}_2(\eta_2)\,d\xi_1\,d\eta_1\,d\xi\,dt \\
&\quad = \iiint \delta\big(P(\eta_1,\xi_1,\xi)\big)\phi_l(\xi_1)\hat{\psi}_1(\xi_1)\phi_m(\xi_2)\hat{\psi}_2(\xi_2)\phi_l(\eta_1)\bar{\hat{\psi}}_1(\eta_1) \\
&\qquad\qquad \times \phi_m(\eta_2)\bar{\hat{\psi}}_2(\eta_2)\,d\xi\,d\xi_1\,d\eta_1,
\end{aligned}$$

where $\eta_2 = \xi - \eta_1$, $\xi_2 = \xi - \xi_1$, and

$$P(\eta_1,\xi_1,\xi) := |\eta_1|^2 \mp \big(|\eta_2|^2+1\big)^{\frac{1}{2}} - |\xi_1|^2 \pm \big(|\xi_2|^2+1\big)^{\frac{1}{2}}.$$

Put $\tilde{\phi}_l = \phi_{l-1} + \phi_l + \phi_{l+1}$ and define

$$G(\xi_1,\eta_1,\xi) := \phi_l(\xi_1)\hat{\psi}_1(\xi_1)\phi_m(\xi_2)\hat{\psi}_2(\xi_2)\tilde{\phi}_l(\eta_1)\tilde{\phi}_m(\eta_2).$$

Then, since $\phi_l(\xi_1)\tilde{\phi}_l(\xi_1) = \phi_l(\xi_1)$, we have

$$\begin{aligned}
&\iiint \delta\big(P(\eta_1,\xi_1,\xi)\big)\phi_l(\xi_1)\hat{\psi}_1(\xi_1)\phi_m(\xi_2)\hat{\psi}_2(\xi_2)\phi_l(\eta_1)\bar{\hat{\psi}}_1(\eta_1) \\
&\qquad \times \phi_m(\eta_2)\bar{\hat{\psi}}_2(\eta_2)\,d\xi_1\,d\xi\,d\eta_1 \\
&\quad = \iiint \delta\big(P(\eta_1,\xi_1,\xi)\big)G(\xi_1,\eta_1,\xi)\overline{G(\eta_1,\xi_1,\xi)}\,d\xi\,d\xi_1\,d\eta_1 \\
&\quad \leqslant I_1^{1/2} I_2^{1/2},
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \iiint \delta\big(P(\eta_1,\xi_1,\xi)\big)\big|G(\xi_1,\eta_1,\xi)\big|^2\,d\xi\,d\xi_1\,d\eta_1, \\
I_2 &:= \iiint \delta\big(P(\eta_1,\xi_1,\xi)\big)\big|G(\eta_1,\xi_1,\xi)\big|^2\,d\xi\,d\xi_1\,d\eta_1.
\end{aligned}$$

Since $\delta(P(\xi_1,\eta_1,\xi)) = \delta(-P(\eta_1,\xi_1,\xi)) = \delta(P(\eta_1,\xi_1,\xi))$, we see that $I_1 = I_2$. At the same time, by Lemma 2.1 and the fact that the estimate

$$\big|\nabla_{\eta_1} P(\eta_1,\xi_1,\xi)\big| = \left|2\eta_1 \mp \frac{\eta_1 - \xi}{(|\eta_1 - \xi|^2 + 1)^{\frac{1}{2}}}\right| \geqslant 2|\eta_1| - 1 \geqslant 2^{l-5}$$

holds on the support of $\tilde{\phi}_l$ when $l \geqslant 2$ (because $\operatorname{supp}\tilde{\phi}_l \subset \{\frac{3}{5} \times 2^{l-2} \leqslant |\eta_1| \leqslant \frac{5}{3} \times 2^{l+1}\}$), we have

$$\begin{aligned}
&\int \delta\big(P(\eta_1,\xi_1,\xi)\big)\tilde{\phi}_l(\eta_1)^2\tilde{\phi}_m(\eta_2)^2\,d\eta_1 \leqslant \int\limits_{P(\eta_1,\xi_1,\xi)=0} \frac{\tilde{\phi}_l(\eta_1)^2\tilde{\phi}_m(\eta_2)^2}{|\nabla_{\eta_1} P(\eta_1,\xi_1,\xi)|}\,dS_{\eta_1} \\
&\quad \leqslant c2^{-l+5} \int\limits_{P(\eta_1,\xi_1,\xi)=0,\ |\xi-\eta_1|\leqslant 2^{m+2}} dS_{\eta_1} \leqslant c2^{m-l}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
I_1 &:= \iint \big|\phi_l(\xi_1)\hat{\psi}_1(\xi_1)\phi_m(\xi_2)\hat{\psi}_2(\xi_2)\big|^2\,d\xi_1\,d\xi \\
&\qquad \times \int \delta\big(P(\eta_1,\xi_1,\xi)\big)\tilde{\phi}_l(\eta_1)^2\tilde{\phi}_m(\eta_2)^2\,d\eta_1 \\
&\leqslant c2^{m-l} \iint \big|\phi_l(\xi_1)\hat{\psi}_1(\xi_1)\phi_m(\xi_2)\hat{\psi}_2(\xi_2)\big|^2\,d\xi_1\,d\xi \\
&\leqslant c2^{m-l}\|\Delta_l\psi_1\|_{L_x^2}^2\|\Delta_m\psi_2\|_{L_x^2}^2.
\end{aligned}$$

Case 2: $l \leqslant 1$

$$\begin{aligned}\left\|U(t)\Delta_l\psi_1 U_\pm(t)\Delta_m\psi_2\right\|_{L^2_{t,x}} &\leqslant \left\|U(t)\Delta_l\psi_1\right\|_{L^4_{t,x}}\left\|U_\pm(t)\Delta_m\psi_2\right\|_{L^4_{t,x}}\\ &\leqslant c\|\Delta_l\psi_1\|_{L^2_x}\|\Delta_m\psi_2\|_{H_x^{(1/2),2}} \leqslant c2^{\frac{m}{2}}\|\Delta_l\psi_1\|_{L^2_x}\|\Delta_m\psi_2\|_{L^2_x},\end{aligned}$$

where we use Strichartz-type estimate with $\gamma=\rho=4$ for the Schrödinger equation and with $\gamma=\rho=4$, $\theta=1$, $\mu=\frac{1}{2}$ for the Klein–Gordon equation. Combining the two cases we get the desired results. □

Lemma 2.4. *For $0\leqslant s<\frac{1}{2}$, then the following estimate holds*:

$$\left\|U(t)uU_\pm(t)v\right\|_{L^2_t(\mathbb{R},H^s_x(\mathbb{R}^2))} \leqslant c\|u\|_{L^2_x(\mathbb{R}^2)}\|v\|_{H_x^{s+}(\mathbb{R}^2)}.$$

Proof. Since

$$fg=\sum_{j\geqslant 0}\sum_{k\geqslant 0}\Delta_j f\Delta_k g=\sum_{j\geqslant 0}S_j f\Delta_j g+\sum_{j\geqslant 0}\Delta_j f S_{j-1}g$$

and since the support of $\mathcal{F}(S_j f\Delta_j g)$ is contained in $\{\xi, |\xi|<2^{j+2}\}$, we get

$$\left\|U(t)uU_\pm(t)v\right\|_{L^2_t(\mathbb{R},H^s_x(\mathbb{R}^2))} \leqslant cI_1+cI_2,$$

where

$$\begin{aligned}I_1 &:= \sum_{j\geqslant 0}2^{js}\left\|U(t)S_j uU_\pm(t)\Delta_j v\right\|_{L^2_t(\mathbb{R},L^2_x(\mathbb{R}^2))},\\ I_2 &:= \sum_{j\geqslant 0}2^{js}\left\|U(t)\Delta_j uU_\pm(t)S_{j-1} v\right\|_{L^2_t(\mathbb{R},L^2_x(\mathbb{R}^2))}.\end{aligned}$$

In order to estimate I_1, we use Strichartz-type estimate with $\gamma=2+$, $\rho=\infty-$ for the Schrödinger equation and with $\gamma=\infty-$, $\rho=2+$, $\theta=1$, $\mu=2(\frac{1}{2}-\frac{1}{2+})=0+$ for the Klein–Gordon equation and get

$$\begin{aligned}\left\|U(t)fU_\pm(t)g\right\|_{L^2_t(\mathbb{R},L^2_x(\mathbb{R}^2))} &\leqslant \left\|U(t)f\right\|_{L^{2+}_t L^{\infty-}_x}\left\|U_\pm(t)g\right\|_{L^{\infty-}_t L^{2+}_x}\\ &\leqslant c\|f\|_{L^2_x}\|g\|_{H^{0+}_x}.\end{aligned}$$

Hence we get

$$I_1\leqslant c\sum_{j\geqslant 0}2^{js}\|S_j u\|_{L^2_x}\|\Delta_j v\|_{H^{0+}_x}\leqslant c\|u\|_{L^2_x}\sum_{j\geqslant 0}2^{j(s+)}\|v\|_{L^2_x}\leqslant c\|u\|_{L^2_x}\|v\|_{H^{s+}_x}.$$

In order to estimate I_2, we use Lemma 2.3 and the fact that

$$\left|\sum_{j\geqslant 0}\sum_{k\geqslant 0}\gamma_{jk}a_j b_k\right| \leqslant \sqrt{c_1 c_2}\,\big\|\{a_j\}_{j\geqslant 0}\big\|_{l^2}\big\|\{b_j\}_{j\geqslant 0}\big\|_{l^2},$$

where $c_1 := \sup_j \sum_{k\geqslant 0}|\gamma_{jk}|$, $c_2 := \sup_k \sum_{j\geqslant 0}|\gamma_{jk}|$, which is an easy consequence of Schwartz's inequality. Namely, we get

$$I_2 \leqslant c\sum_{j\geqslant 0}\sum_{k\leqslant j-1} 2^{(k-j)(\frac{1}{2}-s)}\|\Delta_j u\|_{L_x^2}2^{ks}\|\Delta_k v\|_{L_x^2} \leqslant c\|u\|_{L_x^2}\|v\|_{H_x^s}, \quad \text{if } s<\frac{1}{2}.$$

This completes the proof. □

Lemma 2.5. *For $0\leqslant s<\frac{1}{2}$, then the following estimate holds,*

$$\|v_1 v_2\|_{L_t^2(\mathbb{R},H_x^s(\mathbb{R}^2))} \leqslant c\|v_1\|_{X^{0,(1/2)+}}\|v_2\|_{X_\pm^{s+,(1/2)+}}.$$

Proof. First we have

$$v_1(t) = U(t)\int e^{it\tau}\big(\mathcal{F}_t U(-t)v_1\big)(\tau)\,d\tau = \int e^{it\tau}U(t)\big(\mathcal{F}_t U(-t)v_1\big)(\tau)\,d\tau,$$

$$v_2(t) = U_\pm(t)\int e^{it\tau}\big(\mathcal{F}_t U_\pm(-t)v_2\big)(\tau)\,d\tau = \int e^{it\tau}U_\pm(t)\big(\mathcal{F}_t U_\pm(-t)v_2\big)(\tau)\,d\tau.$$

Thus by Lemma 2.4, we get

$$\begin{aligned}
&\|v_1 v_2\|_{L_t^2 H_x^s}\\
&\quad= \left\|\iint e^{it\tau_1}U(t)\big(\mathcal{F}_t U(-t)v_1\big)(\tau_1)e^{it\tau_2}U_\pm(t)\big(\mathcal{F}_t U_\pm(-t)v_2\big)(\tau_2)\,d\tau_1\,d\tau_2\right\|_{L_t^2 H_x^s}\\
&\quad\leqslant \iint \big\|U(t)\big(\mathcal{F}_t U(-t)v_1\big)(\tau_1)U_\pm(t)\big(\mathcal{F}_t U_\pm(-t)v_2\big)(\tau_2)\big\|_{L_t^2 H_x^s}\,d\tau_1\,d\tau_2\\
&\quad\leqslant c\iint \big\|\big(\mathcal{F}_t U(-t)v_1\big)(\tau_1)\big\|_{L_x^2}\big\|\big(\mathcal{F}_t U_\pm(-t)v_2\big)(\tau_2)\big\|_{H_x^{s+}}\,d\tau_1\,d\tau_2\\
&\quad\leqslant c\left(\int\langle\tau_1\rangle^{-1-}\,d\tau_1\right)^{\frac{1}{2}}\left(\int\langle\tau_1\rangle^{1+}\big\|\big(\mathcal{F}_t U(-t)v_1\big)(\tau_1)\big\|_{L_x^2}^2\,d\tau_1\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int\langle\tau_2\rangle^{-1-}\,d\tau_2\right)^{\frac{1}{2}}\left(\int\langle\tau_2\rangle^{1+}\big\|\big(\mathcal{F}_t U_\pm(-t)v_2\big)(\tau_2)\big\|_{H_x^{s+}}^2\,d\tau_2\right)^{\frac{1}{2}}\\
&\quad\leqslant c\|v_1\|_{X^{0,(1/2)+}}\|v_2\|_{X_\pm^{s+,(1/2)+}}. \qquad\square
\end{aligned}$$

Corollary 2.1. *The following estimates hold for* $0 \leqslant s < \frac{1}{2}$,

$$\|un\|_{X^{0,-(1/2)-}} \leqslant c\|u\|_{L_t^2(\mathbb{R},H_x^{-s}(\mathbb{R}^2))}\|n\|_{X_{\pm}^{s+,(1/2)+}}.$$

Here u and/or n can be replaced by $\bar{u}$ and/or $\bar{n}$ on the left- and/or right-hand side.

Proof. By Lemma 2.5 the mapping $X^{0,\frac{1}{2}+} \to L_t^2 H_x^s$ defined by $u \mapsto un$ is bounded by $c\|n\|_{X_{\pm}^{s+,(1/2)+}}$. Thus the dual mapping $L_t^2 H_x^{-s} \to X^{0,-\frac{1}{2}-}$ defined by $u \mapsto u\bar{n}$ has the same bound, i.e.

$$\|u\bar{n}\|_{X^{0,-(1/2)-}} \leqslant c\|u\|_{L_t^2 H_x^{-s}}\|n\|_{X_{\pm}^{s+,(1/2)+}}.$$

Because

$$\begin{aligned}
\|\bar{n}\|^2_{X_+^{s+,(1/2)+}} &= \iint \left|\langle \tau + |\xi|\rangle^{\frac{1}{2}+}\langle\xi\rangle^{s+}\hat{\bar{n}}(\xi,\tau)\right|^2 d\xi\, d\tau \\
&= \iint \left|\langle \tau + |\xi|\rangle^{\frac{1}{2}+}\langle\xi\rangle^{s+}\bar{\hat{n}}(-\xi,-\tau)\right|^2 d\xi\, d\tau \\
&= \iint \left|\langle \tau - |\xi|\rangle^{\frac{1}{2}+}\langle\xi\rangle^{s+}\hat{n}(\xi,\tau)\right|^2 d\xi\, d\tau = \|n\|^2_{X_-^{s+,(1/2)+}},
\end{aligned}$$

we can replace n by $\bar{n}$ on the left and/or right side of the claimed inequality. At the same time u may be replaced by $\bar{u}$ on the left- and/or right-hand side. □

The key point is a smoothing estimate for the quadratic nonlinearity in the Schrödinger equation, which is given in the following lemma.

Lemma 2.6. *For $s \geqslant 0$, $0 \leqslant \sigma < \frac{1}{2}$, then the following estimate holds*:

$$\|un_\pm\|_{X^{s,-(1/2)-}_{\tau=-|\xi|^2}} \leqslant c\left(\|u\|_{X^{0,(1/2)+}_{\tau=-|\xi|^2}}\|n_\pm\|_{X_\pm^{s-1,(1/2)+}} + \|u\|_{X^{s-\sigma,0}_{\tau=-|\xi|^2}}\|n_\pm\|_{X_\pm^{\sigma+,(1/2)+}}\right).$$

Here u and/or n can be replaced by $\bar{u}$ and/or $\bar{n}$ on the left- and/or right-hand side.

Proof. We take the scalar product with a function $w \in X^{-s,\frac{1}{2}+}_{\tau=|\xi|^2}$ and estimate

$$\begin{aligned}
&\left|\int (un_\pm)(x_1,t_1)w(x_1,t_1)\,dx_1\,dt_1\right| \\
&\quad = \left|\iint \hat{u}(\xi_2,\tau_2)\hat{n}_\pm(\xi,\tau)\hat{w}(-\xi_1,-\tau_1)\,d\xi_2\,d\tau_2\,d\xi_1\,d\tau_1\right|,
\end{aligned}$$

where $\xi = \xi_1 - \xi_2$, $\tau = \tau_1 - \tau_2$.

We split the integral domain into the parts B_1: $|\xi_1| \leqslant 2|\xi_2|$ and B_2: $|\xi_1| > 2|\xi_2|$.

Estimate of B_1. We use the notation $\sigma_i = \tau_i + |\xi_i|^2$, $i = 1, 2$; $\sigma_\pm = \tau \pm \langle\xi\rangle$ and Corollary 2.1

$$\begin{aligned}
&\left| \iint\limits_{B_1} \hat{u}(\xi_2, \tau_2)\hat{n}_\pm(\xi, \tau)\hat{w}(-\xi_1, -\tau_1)\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1 \right| \\
&\quad = \left| \int \langle\xi_1\rangle^s \langle\sigma_1\rangle^{-\frac{1}{2}-} \int\limits_{B_1} \hat{u}(\xi_2, \tau_2)\hat{n}_\pm(\xi, \tau)\, d\xi_2\, d\tau_2 \right. \\
&\qquad \left. \times \langle\xi_1\rangle^{-s} \langle\sigma_1\rangle^{\frac{1}{2}+} \hat{w}(-\xi_1, -\tau_1)\, d\xi_1\, d\tau_1 \right| \\
&\quad \leqslant \left\| \langle\xi_1\rangle^s \langle\sigma_1\rangle^{-\frac{1}{2}-} \int\limits_{B_1} \hat{u}(\xi_2, \tau_2)\hat{n}_\pm(\xi, \tau)\, d\xi_2\, d\tau_2 \right\|_{L^2_{\xi_1\tau_1}} \times \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} \\
&\quad \leqslant c \left\| \langle\sigma_1\rangle^{-\frac{1}{2}-} \int\limits_{B_1} \langle\xi_2\rangle^s \hat{u}(\xi_2, \tau_2)\hat{n}_\pm(\xi, \tau)\, d\xi_2\, d\tau_2 \right\|_{L^2_{\xi_1\tau_1}} \times \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} \\
&\quad \leqslant c \left\| \langle\sigma_1\rangle^{-\frac{1}{2}-} \int \langle\xi_2\rangle^s \left|\hat{u}(\xi_2, \tau_2)\hat{n}_\pm(\xi, \tau)\right| d\xi_2\, d\tau_2 \right\|_{L^2_{\xi_1\tau_1}} \times \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} \\
&\quad = c \left\| \left(A^s \mathcal{F}^{-1}|\hat{u}|\right)\left(\mathcal{F}^{-1}|\hat{n}_\pm|\right) \right\|_{X^{0,(1/2)-}_{\tau=-|\xi|^2}} \times \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} \\
&\quad \leqslant c \left\| A^s \mathcal{F}^{-1}|\hat{u}| \right\|_{L^2_t H^{-\sigma}_x} \left\| \mathcal{F}^{-1}|\hat{n}_\pm| \right\|_{X^{\sigma+,(1/2)+}_\pm} \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} \\
&\quad = c \left\| \mathcal{F}^{-1}|\hat{u}| \right\|_{L^2_t H^{s-\sigma}_x} \left\| \mathcal{F}^{-1}|\hat{n}_\pm| \right\|_{X^{\sigma+,(1/2)+}_\pm} \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} \\
&\quad = \|u\|_{X^{s-\sigma,0}_{\tau=-|\xi|^2}} \|n_\pm\|_{X^{\sigma+,(1/2)+}_\pm} \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}};
\end{aligned}$$

Estimate of B_2. Define

$$\begin{aligned}
\hat{v}(\xi, \tau) &:= \langle\xi\rangle^{s-1} \langle\sigma_\pm\rangle^{\frac{1}{2}+} \hat{n}_\pm(\xi, \tau), \\
\hat{v}_1(\xi_1, \tau_1) &:= \langle\xi_1\rangle^{-s} \langle\sigma_1\rangle^{\frac{1}{2}+} \hat{w}(-\xi_1, -\tau_1), \\
\hat{v}_2(\xi_2, \tau_2) &:= \langle\sigma_2\rangle^{\frac{1}{2}+} \hat{u}(\xi_2, \tau_2),
\end{aligned}$$

so that

$$\|n_\pm\|_{X^{s-1,(1/2)+}_\pm} = \|v\|_{L^2_{xt}}, \qquad \|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}} = \|v_1\|_{L^2_{xt}}, \qquad \|u\|_{X^{0,(1/2)+}_{\tau=-|\xi|^2}} = \|v_2\|_{L^2_{xt}}.$$

In B_2, we have

$$\frac{1}{2}|\xi_1| \leqslant |\xi| \leqslant \frac{3}{2}|\xi_1|, \qquad \frac{1}{2}\langle\xi_1\rangle \leqslant \langle\xi\rangle \leqslant \frac{3}{2}\langle\xi_1\rangle,$$

and

$$\begin{aligned} z_\pm &:= |\xi_1|^2 - |\xi_2|^2 \mp \langle\xi\rangle = (\sigma_1 - \tau_1) - (\sigma_2 - \tau_2) + (\tau - \sigma_\pm) \\ &= \sigma_1 - \sigma_2 - \sigma_\pm + \tau_2 - \tau_1 + \tau = \sigma_1 - \sigma_2 - \sigma_\pm. \end{aligned}$$

Thus

$$\begin{aligned} &\big||\xi_1|^2 - |\xi_2|^2 \mp \langle\xi\rangle\big| \\ &\quad \geqslant |\xi_1|^2 - |\xi_2|^2 - \langle\xi\rangle \geqslant |\xi_1|^2 - \frac{1}{4}|\xi_1|^2 - \big(|\xi_1 - \xi_2| + 1\big) \\ &\quad \geqslant \frac{3}{4}|\xi_1|^2 - \left(\frac{3}{2}|\xi_1| + 1\right) \geqslant \frac{3}{4}|\xi_1|^2 - \left(\frac{3}{8}|\xi_1|^2 + \frac{5}{2}\right) = \frac{3}{8}|\xi_1|^2 - \frac{5}{2}. \end{aligned}$$

This implies

$$\frac{3}{8}|\xi_1|^2 \leqslant \big||\xi_1|^2 - |\xi_2|^2 \mp \langle\xi\rangle\big| + \frac{5}{2} \leqslant |\sigma_1| + |\sigma_2| + |\sigma_\pm| + \frac{5}{2},$$

and

$$\langle\xi_1\rangle^{\frac{1}{2}} \leqslant c\big(\langle\sigma_1\rangle^{\frac{1}{4}} + \langle\sigma_2\rangle^{\frac{1}{4}} + \langle\sigma_\pm\rangle^{\frac{1}{4}}\big).$$

Therefore we get

$$\begin{aligned} &\left|\iint_{B_2} \hat{u}(\xi_2, \tau_2)\hat{n}_\pm(\xi, \tau)\hat{w}(-\xi_1, -\tau_1)\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1\right| \\ &\quad = \left|\iint_{B_2} \frac{\hat{v}_2\hat{v}\hat{v}_1\langle\xi_1\rangle^s}{\langle\sigma_2\rangle^{\frac{1}{2}+}\langle\sigma_\pm\rangle^{\frac{1}{2}+}\langle\xi\rangle^{s-1}\langle\sigma_1\rangle^{\frac{1}{2}+}}\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1\right| \\ &\quad \leqslant \iint \frac{|\hat{v}_2\hat{v}\hat{v}_1|\langle\xi_1\rangle}{\langle\sigma_2\rangle^{\frac{1}{2}+}\langle\sigma_\pm\rangle^{\frac{1}{2}+}\langle\sigma_1\rangle^{\frac{1}{2}+}}\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1 \\ &\quad \leqslant c\iint \frac{|\hat{v}_2\hat{v}\hat{v}_1|(\langle\sigma_1\rangle^{\frac{1}{2}} + \langle\sigma_2\rangle^{\frac{1}{2}} + \langle\sigma_\pm\rangle^{\frac{1}{2}})}{\langle\sigma_2\rangle^{\frac{1}{2}+}\langle\sigma_\pm\rangle^{\frac{1}{2}+}\langle\sigma_1\rangle^{\frac{1}{2}+}}\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1 \\ &\quad \leqslant c(I_1 + I_2 + I_3), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \iint \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle\sigma_\pm\rangle^{0+}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{\frac{1}{2}+}}\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1, \\ I_2 &= \iint \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle\sigma_\pm\rangle^{\frac{1}{2}+}\langle\sigma_1\rangle^{0+}\langle\sigma_2\rangle^{\frac{1}{2}+}}\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1, \\ I_3 &= \iint \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle\sigma_\pm\rangle^{\frac{1}{2}+}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{0+}}\, d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1. \end{aligned}$$

Our aim is to estimate each of these integrals by

$$c\|v\|_{L^2_{xt}}\|v_1\|_{L^2_{xt}}\|v_2\|_{L^2_{xt}} = c\|n_\pm\|_{X_\pm^{s-1,(1/2)+}}\|w\|_{X^{-s,(1/2)+}_{\tau=|\xi|^2}}\|u\|_{X^{0,(1/2)+}_{\tau=-|\xi|^2}}.$$

These estimates together with the estimate of B_1 imply the claimed results. I_2 and I_3 can be treated in the same way, so we only deal with I_1 and I_2.

Estimate of I_1:

By Hölder inequality we get

$$\begin{aligned} I_1 &\leqslant \big\|\mathcal{F}^{-1}\big(\langle\sigma_\pm\rangle^{0-}|\hat{v}|\big)\big\|_{L^2_{tx}} \prod_{i=1}^{2} \big\|\mathcal{F}^{-1}\big(\langle\sigma_i\rangle^{-\frac{1}{2}-}|\hat{v}_i|\big)\big\|_{L^4_{tx}} \\ &\leqslant c\big\|\mathcal{F}^{-1}\big(\langle\sigma_\pm\rangle^{0-}|\hat{v}|\big)\big\|_{X^{0,0}_\pm} \prod_{i=1}^{2} \big\|\mathcal{F}^{-1}\big(\langle\sigma_i\rangle^{-\frac{1}{2}-}|\hat{v}_i|\big)\big\|_{X^{0,(1/2)+}} \\ &\leqslant c\|v\|_{L^2_{tx}}\|v_1\|_{L^2_{tx}}\|v_2\|_{L^2_{tx}}, \end{aligned}$$

where we use inequality (1.7) with $\gamma = \rho = 4$ for the Schrödinger equation.

Estimate of I_2. We have

$$\begin{aligned} I_2 \leqslant{}& \big\|\mathcal{F}^{-1}\big(\langle\sigma_\pm\rangle^{-\frac{1}{2}-\epsilon}|\hat{v}|\big)\big\|_{L_t^{\infty-}L_x^2}\big\|\mathcal{F}^{-1}\big(\langle\sigma_2\rangle^{-\frac{1}{2}-\epsilon}|\hat{v}_2|\big)\big\|_{L_t^{2+}L_x^{\infty-}} \\ &\times \big\|\mathcal{F}^{-1}\big(\langle\sigma_1\rangle^{0-\epsilon}|\hat{v}_1|\big)\big\|_{L_t^{2+}L_x^{2+}}. \end{aligned}$$

Now we use the inequality $\|w\|_{L_t^\infty L_x^2} \leqslant c\|w\|_{X_\pm^{0,(1/2)+}}$ for the Klein–Gordon equation. Interpolating with $\|w\|_{L^2_{tx}} = \|w\|_{X^{0,0}_\pm}$, we get $\|w\|_{L_t^{\infty-}L_x^2} \leqslant c\|w\|_{X_\pm^{0,(1/2)+}}$. This implies

$$\big\|\mathcal{F}^{-1}\big(\langle\sigma_\pm\rangle^{-\frac{1}{2}-\epsilon}|\hat{v}|\big)\big\|_{L_t^{\infty-}L_x^2} \leqslant c\big\|\mathcal{F}^{-1}\big(\langle\sigma_\pm\rangle^{-\frac{1}{2}-\epsilon}|\hat{v}|\big)\big\|_{X_\pm^{0,(1/2)+}} \leqslant c\|v\|_{X^{0,0}_\pm} = c\|v\|_{L^2_{tx}}.$$

Using Strichartz-type estimate for the Schrödinger equation, we get from (1.7)

$$\big\|\mathcal{F}^{-1}\big(\langle\sigma_2\rangle^{-\frac{1}{2}-\epsilon}|\hat{v}_2|\big)\big\|_{L_t^{2+}L_x^{\infty-}} \leqslant c\big\|\mathcal{F}^{-1}\big(\langle\sigma_2\rangle^{-\frac{1}{2}-\epsilon}|\hat{v}_2|\big)\big\|_{X^{0,(1/2)+}} \leqslant c\|v_2\|_{L^2_{tx}}.$$

Finally, by virtue of (1.7) and the trivial identity $\|w\|_{L^2_{tx}} = \|w\|_{X^{0,0}_\pm}$, we get by interpolation

$$\big\|\mathcal{F}^{-1}\big(\langle\sigma_1\rangle^{-\epsilon}|\hat{v}_1|\big)\big\|_{L_t^{2+}L_x^{2+}} \leqslant c\big\|\mathcal{F}^{-1}\big(\langle\sigma_1\rangle^{-\epsilon}|\hat{v}_1|\big)\big\|_{X^{0,0+}} \leqslant c\|v_1\|_{X^{0,0}} \leqslant c\|v_1\|_{L^2_{tx}},$$

which completes the proof. $\square$

3. Local existence and uniqueness

In this section, we transform system (1.1) into an equivalent system of first order in t in the usual way and prove the local existence and uniqueness of system (1.1).

Define

$$\phi_{\pm} := \phi \pm i A^{-1} \phi_t,$$

we have

$$\phi = \frac{1}{2}(\phi_+ + \phi_-), \qquad \phi_t = -\frac{i}{2} A(\phi_+ - \phi_-),$$

and the equivalent system are

$$\begin{cases} i\psi_t + \Delta\psi = -\frac{1}{2}(\phi_+ + \phi_-)\psi + |\psi|^2\psi, \\ i\phi_{\pm t} \mp A\phi_{\pm} = \mp A^{-1}(|\psi|^2) \pm \frac{1}{8} A^{-1}(\phi_+ + \phi_-)^3, \\ \psi(0) = \psi_0, \\ \phi_{\pm}(0) = \phi_0 \pm i A^{-1}\phi_1 =: \phi_{0\pm}. \end{cases} \tag{3.1}$$

The corresponding system of integral equations are as follows:

$$\begin{cases} \psi(t) = U(t)\psi_0 + i\int_0^t U(t-s)[\frac{1}{2}(\phi_+(s) + \phi_-(s))\psi(s) - |\psi(s)|^2\psi(s)]\,ds, \\ \phi_{\pm}(t) = U_{\mp}(t)\phi_{0\pm} \pm i\int_0^t U_{\mp}(t-s)A^{-1}[(|\psi(s)|^2) - \frac{1}{8}(\phi_+(s) + \phi_-(s))^3]\,ds. \end{cases} \tag{3.2}$$

Remark. We always assume $t \in [0, |I|]$, in this case we could, whenever helpful, place a factor $\chi_1(t)$ in front of the first terms on the right-hand side and a factor $\chi_{|I|}(t)$ in front of any of the integrals without changing the equations at all. Here $\chi \in C_0^{\infty}(\mathbb{R})$ is a nonnegative cut-off function with $\chi(t) = 0$ if $|t| \geqslant 2$, $\chi(t) = 1$ if $|t| \leqslant 1$ and $\chi_\delta(t) = \chi(\frac{t}{\delta})$.

The quadratic coupling estimates follow from the corresponding results for the Zakharov system in [15] as follows.

Lemma 3.1. *Let $m \geqslant 0$, $s < m+1$. Then the following estimates hold for $\epsilon > 0$*

$$\begin{aligned} \|\psi\phi_{\pm}\|_{X^{s,-(1/2)+}([0,T])} &\leqslant cT^{\frac{1}{4}+}\|\psi\|_{X^{s,(1/2)+}([0,T])}\|\phi_{\pm}\|_{X_{\pm}^{m,(1/2)+}([0,T])}, \\ \|\psi\phi_{\pm}\|_{X^{0,-(1/4)-\epsilon}([0,T])} &\leqslant cT^{0+}\|\psi\|_{X^{0,(1/2)+\epsilon}([0,T])}\|\phi_{\pm}\|_{X_{\pm}^{0,(1/2)+\epsilon}([0,T])}. \end{aligned} \tag{3.3}$$

Proof. By [15, Lemma 3.4], we get the first inequality with $b_0 = \frac{1}{2}+$, $b_1 = \frac{1}{2}+$, $b = \frac{1}{2}+$, $c_0 = \frac{1}{2}-$, $c_1 = \frac{1}{2}-$, $l = m$, $k = s$, $\theta = \frac{1}{4}+$, and the second inequality with $b_0 = \frac{1}{2}+\epsilon$, $b_1 = \frac{1}{2}+\epsilon$, $b = \frac{1}{2}+\epsilon$, $c_0 = \frac{1}{2}-$, $c_1 = \frac{1}{4}+\epsilon$, $l = 0$, $k = 0$, $\theta = 0+$. □

Lemma 3.2. *Let $s \geqslant 0$, $2s \geqslant m-1$, $s > m-2$. Then the following estimate holds*:

$$\big\| |\psi|^2 \big\|_{X_\pm^{m-1,-(1/2)+}} \leqslant cT^{\frac{1}{4}+} \|\psi\|^2_{X^{s,(1/2)+}}. \tag{3.4}$$

Proof. It follows from [15, Lemma 3.5] with $b_0 = \frac{1}{2}+$, $b_1 = \frac{1}{2}+$, $\bar{c}_0 = \frac{1}{2}-$, $c = \frac{1}{2}-$, $l = m-2$, $k = s$, $\theta = \frac{1}{4}+$.

The cubic autointeraction estimates we need for the global solution of system (1.1) are the following lemmas.

First by (1.9), the so-called fractional Leibniz rules, Sobolev inequality and (1.7), we get the following inequalities:

(a1) Let $0 \leqslant 2\alpha = \sigma_1 + \sigma_2 + \sigma_3 \leqslant 2$, $0 \leqslant \sigma_i < 1$ $(i = 1, 2, 3)$, $0 \leqslant s \leqslant 1$. Then

$$\|u_1u_2u_3\|_{X^{0,-(1/2)+}(I)} \leqslant c\|u_1u_2u_3\|_{L_t^{1+}(I,L_x^2(\mathbb{R}^2))} \leqslant c|I|^{\alpha-} \prod_{i=1}^{3} \|u_i\|_{X^{\sigma_i,(1/2)+}(I)}.$$

(a2) Let $0 \leqslant s \leqslant 1$, $0 \leqslant 2\alpha = \sigma_{i1} + \sigma_{i2} + \sigma_{i3} \leqslant 2$, $0 \leqslant \sigma_{ij} < 1$ $(i, j = 1, 2, 3)$. Then

$$\begin{aligned}
\|u_1u_2u_3\|_{X^{s,-(1/2)+}(I)} &\leqslant c\|u_1u_2u_3\|_{L_t^{1+}(I,H_x^s(\mathbb{R}^2))} \\
&\leqslant c|I|^{\alpha-} \|u_1\|_{X^{s+\sigma_{11},(1/2)+}(I)} \prod_{i=2,3} \|u_i\|_{X^{\sigma_{1i},(1/2)+}(I)} \\
&\quad + c|I|^{\alpha-} \|u_2\|_{X^{s+\sigma_{22},(1/2)+}(I)} \prod_{i=1,3} \|u_i\|_{X^{\sigma_{2i},(1/2)+}(I)} \\
&\quad + c|I|^{\alpha-} \|u_3\|_{X^{s+\sigma_{33},(1/2)+}(I)} \prod_{i=1,2} \|u_i\|_{X^{\sigma_{3i},(1/2)+}(I)}.
\end{aligned}$$

$$\|u_1u_2\bar{u}_3\|_{X^{0,-(1/2)+}(I)}, \quad \|u_1\bar{u}_2\bar{u}_3\|_{X^{0,-(1/2)+}(I)}, \quad \|u_1u_2\bar{u}_3\|_{X^{s,-(1/2)+}(I)},$$
$$\|u_1\bar{u}_2\bar{u}_3\|_{X^{s,-(1/2)+}(I)}$$

are estimated in the same way.

(a3) Let $0 \leqslant m \leqslant 1$, $0 \leqslant m - 2(1-\alpha) = \sigma_1 + \sigma_2$, $0 \leqslant \sigma_i < 1$ for $i = 1, 2$. Then

$$\|u_1u_2\|_{X_\pm^{m-1,-(1/2)+}(I)} \leqslant c|I|^{\alpha-} \|u_1\|_{X^{\sigma_1,(1/2)+}(I)} \|u_2\|_{X^{\sigma_2,(1/2)+}(I)}.$$

Lemma 3.3. *Let $\frac{1}{2} \leqslant s \leqslant 1$. Then the following estimate holds*:

$$\big\| |\psi|^2\psi \big\|_{X^{s,-(1/2)+}([0,T])} \leqslant cT^{\frac{1}{2}-} \|\psi\|^3_{X^{s,(1/2)+}([0,T])}.$$

Proof. Take $\alpha = \frac{1}{2}$, $\sigma_{ij} = \frac{1}{2}$ if $i \neq j$ and $\sigma_{ii} = 0$. Then the estimate of $\||\psi|^2\psi\|_{X^{s,-(1/2)+}([0,T])}$ follows directly from (a2). □

Next by (1.13), Sobolev inequality and (1.11), we get the following inequalities:

(b1) Let $1/2 \leqslant \alpha \leqslant 1$, $\sigma = (1+\alpha)/3$. Then

$$\|v_1 v_2 v_3\|_{X_\pm^{0,-(1/2)+}(I)} \leqslant c\|v_1 v_2 v_3\|_{L_t^{1+}(I,L_x^2(\mathbb{R}^2))} \leqslant c|I|^{\alpha-}\prod_{i=1}^{3}\|v_i\|_{X_\pm^{\sigma,(1/2)+}(I)}.$$

Lemma 3.4. *Let $\frac{1}{2} \leqslant m \leqslant 1$. Then the following estimate holds*

$$\left\|(\phi_\pm)^3\right\|_{X_\pm^{m-1,-(1/2)+}([0,T])} \leqslant cT^{\frac{1}{2}-}\|\phi_\pm\|^3_{X_\pm^{m,(1/2)+}([0,T])}.$$

Proof. Take $\alpha = 1/2$. Then the estimate of $\|(\phi_\pm)^3\|_{X_\pm^{m-1,-(1/2)+}([0,T])}$ follows directly from (b1). $\square$

By Lemmas 3.1–3.4, we at once have the following local well-posedness we need for the global solution without finite energy of system (1.1).

Theorem 3.1. *Let $\frac{1}{2} \leqslant s \leqslant 1$, $\frac{1}{2} \leqslant m \leqslant 1$. Then system* (1.1) *with initial data*

$$(\psi_0, \phi_0, \phi_1) \in H^s(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2)$$

has a unique local solution

$$\begin{aligned}(\psi, \phi, \phi_t) \in\ & X^{s,\frac{1}{2}+}[0,T] \times \left(X_+^{m,\frac{1}{2}+}[0,T] + X_-^{m,\frac{1}{2}+}[0,T]\right)\\ & \times \left(X_+^{m-1,\frac{1}{2}+}[0,T] + X_-^{m-1,\frac{1}{2}+}[0,T]\right).\end{aligned}$$

We also have

$$(\psi, \phi, \phi_t) \in C\big([0,T], H^s(\mathbb{R}^2)\big) \times C\big([0,T], H^m(\mathbb{R}^2)\big) \times C\big([0,T], H^{m-1}(\mathbb{R}^2)\big).$$

4. Energy bounds and decomposition of data

In this section, we split the data into sums $\psi_0 = \psi_{01} + \psi_{02}$, $\phi_0 = \phi_{01} + \phi_{02}$, $\phi_1 = \phi_{11} + \phi_{12}$, where the low frequency parts $(\psi_{01}, \phi_{01}, \phi_{11})$ are regular with large norms and the high frequency parts $(\psi_{02}, \phi_{02}, \phi_{12})$ are just in $H^s(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2)$ with small L^2-norms. At the same time, we give the energy bounds on the global solution $(\tilde{\psi}, \tilde{\phi})$ of (1.1) with data $(\psi_{01}, \phi_{01}, \phi_{11})$.

From the conservation laws (1.2) and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}|\psi|^2\phi\,dx\right| &\leqslant \|\phi\|_{L^6}\|\psi\|^2_{L^{12/5}} \leqslant c\|A\phi\|\|\nabla\psi\|^{\frac{1}{3}}\|\psi\|^{\frac{5}{3}}\\ &\leqslant \frac{1}{4}\|A\phi\|^2+\frac{1}{3}\|\nabla\psi\|^2+c_1\|\psi\|^5,\\ \int_{\mathbb{R}^2}|\psi|^4\,dx &\leqslant c_2\|\psi\|^2\|\nabla\psi\|^2 \leqslant \frac{c_2}{2}\big[\|\psi\|^4+\|\nabla\psi\|^4\big],\\ \int_{\mathbb{R}^2}|\phi|^4\,dx &\leqslant c_3\|A\phi\|^4,\end{aligned}$$

where $c_1=\frac{2}{3}c^3$. This implies

$$\begin{aligned}&\big\|\nabla\psi(t)\big\|^2+\frac{1}{2}\big(\big\|A\phi(t)\big\|^2+\big\|\phi_t(t)\big\|^2\big)+\frac{1}{2}\int_{\mathbb{R}^2}|\psi|^4\,dx+\frac{1}{4}\int_{\mathbb{R}^2}|\phi|^4\,dx\\ &\quad\leqslant E+\frac{1}{4}\big\|A\phi(t)\big\|^2+\frac{1}{3}\big\|\nabla\psi(t)\big\|^2+c_1M^5.\end{aligned}$$

Consequently

$$\big\|\nabla\psi(t)\big\|^2\leqslant\frac{3}{2}\big(E+c_1M^5\big),\qquad \big\|A\phi(t)\big\|^2+\big\|\phi_t(t)\big\|^2\leqslant 4\big(E+c_1M^5\big).\tag{4.1}$$

We also have

$$\begin{aligned}&E(\psi,\phi,\phi_t,t)\\ &\quad\leqslant\big\|\nabla\psi(t)\big\|^2+\frac{1}{2}\big(\big\|A\phi(t)\big\|^2+\big\|\phi_t(t)\big\|^2\big)+\left|\int_{\mathbb{R}^2}\big|\psi(t)\big|^2\phi(t)\,dx\right|\\ &\qquad+\frac{1}{2}\int_{\mathbb{R}^2}|\psi|^4\,dx+\frac{1}{4}\int_{\mathbb{R}^2}|\phi|^4\,dx\\ &\quad\leqslant\frac{c_2}{4}\|\nabla\psi\|^4+\frac{4}{3}\big\|\nabla\psi(t)\big\|^2+\frac{c_3}{4}\|A\phi\|^4+\frac{3}{4}\big(\big\|A\phi(t)\big\|^2+\big\|\phi_t(t)\big\|^2\big)\\ &\qquad+c_1M^5+\frac{c_2}{4}M^4.\end{aligned}\tag{4.2}$$

Consider now data

$$\psi_0\in H^s\big(\mathbb{R}^2\big),\quad \phi_0\in H^m\big(\mathbb{R}^2\big),\quad \phi_1\in H^{m-1}\big(\mathbb{R}^2\big)$$

with $0\leqslant s,m\leqslant 1$.

We split these data into sums as follows:

$$\begin{cases} \psi_0 = \psi_{01} + \psi_{02}, \\ \phi_0 = \phi_{01} + \phi_{02}, \\ \phi_1 = \phi_{11} + \phi_{12}, \end{cases}$$

where, for $N \geqslant 1$,

$$\psi_{01} := (2\pi)^{-\frac{n}{2}} \int\limits_{|\xi| \leqslant N} e^{i\langle x,\xi\rangle} \hat{\psi}_0(\xi)\, d\xi$$

and ϕ_{01}, ϕ_{11} are defined in the same way.

We easily show that

$$\begin{cases} \|\psi_{01}\| \leqslant \|\psi_0\| = M, \\ \|\psi_{01}\|_{H^l} \leqslant cN^{l-s} \qquad \text{for } l \geqslant s, \\ \|\psi_{02}\|_{H^l} \leqslant cN^{l-s} \qquad \text{for } l \leqslant s; \end{cases}$$

and similarly

$$\begin{cases} \|\phi_{01}\| \leqslant \|\phi_0\|, \\ \|\phi_{01}\|_{H^l} \leqslant cN^{l-m} \quad \text{for } l \geqslant m, \\ \|\phi_{02}\|_{H^l} \leqslant cN^{l-m} \quad \text{for } l \leqslant m; \end{cases} \qquad \begin{cases} \|\phi_{11}\|_{H^{l-1}} \leqslant cN^{l-m} \quad \text{for } l \geqslant m, \\ \|\phi_{12}\|_{H^{l-1}} \leqslant cN^{l-m} \quad \text{for } l \leqslant m. \end{cases}$$

Thus we have the following global bounds for the solution $(\tilde{\psi}, \tilde{\phi})$ of (1.1) with data $(\psi_{01}, \phi_{01}, \phi_{11})$, similar to [1, Theorem 3], by (4.2),

$$\begin{aligned} E(\tilde{\psi}, \tilde{\phi}, \tilde{\phi}_t) &\leqslant \frac{c_2}{4}\|\nabla\psi_{01}\|^4 + \frac{4}{3}\|\nabla\psi_{01}\|^2 + \frac{c_3}{4}\|A\phi_{01}\|^4 \\ &\quad + \frac{3}{4}\big(\|A\phi_{01}\|^2 + \|\phi_{11}\|^2\big) + c_1\|\psi_{01}\|^5 + \frac{c_2}{4}\|\psi_{01}\|^4 \\ &\leqslant \frac{\bar{c}}{2}\big(N^{4(1-s)} + N^{4(1-m)}\big) \leqslant \bar{c}N^{4(1-s\wedge m)}, \end{aligned} \tag{4.3}$$

where $s \wedge m = \min(s, m)$ and N sufficient large, thus by (1.2) and (4.1),

$$\begin{gathered} \|\tilde{\psi}(t)\| \leqslant M, \\ \|\nabla\tilde{\psi}(t)\| + \|A\tilde{\phi}(t)\| + \|\tilde{\phi}_t(t)\| \leqslant \hat{c}N^{2(1-s\wedge m)}. \end{gathered} \tag{4.4}$$

The corresponding global solution $(\tilde{\psi}, \tilde{\phi}_\pm)$ of (3.1) with data ψ_{01}, $\phi_{0\pm1} := \phi_{01} \pm iA^{-1}\phi_{11}$ therefore satisfies

$$\|\nabla\tilde{\psi}(t)\| + \|A\tilde{\phi}_\pm(t)\| \leqslant \hat{c}N^{2(1-s\wedge m)}, \tag{4.5}$$

where $\hat{c}$ depends essentially only on $\bar{c}(\|\psi_0\|_{H^s}, \|\phi_0\|_{H^m}, \|\phi_1\|_{H^{m-1,2}})$ and M.

5. Further bounds for the regular part

In this section, we get further bounds on the solution $(\tilde{\psi}, \tilde{\phi}_+, \tilde{\phi}_-)$ of the corresponding first order system with data $(\psi_{01}, \phi_{01}, \phi_{11})$ on a suitable time interval I by virtue of the conservation laws and Strichartz-type estimates.

Consider the system of integral equations (3.2) with $(\psi_0, \phi_{0\pm})$ replaced by $(\psi_{01}, \phi_{0\pm 1})$ and $\psi, \phi_\pm$ by $(\tilde{\psi}, \tilde{\phi}_\pm)$. Here $\phi_{0\pm 1} := \phi_{01} \pm iA^{-1}\phi_{11}$.

We assume $|I| \leqslant 1$. First we estimate $\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)}$. Applying (1.10) to (3.2), we get by the remarks after (3.2) and (1.5), (1.6)

$$\begin{aligned}
&\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)} \\
&\quad\leqslant c\left(\|\psi_{01}\|_{H^1} + \left\| \int_0^t U(t-s)\left[\frac{1}{2}\big(\tilde{\phi}_+(s) + \tilde{\phi}_-(s)\big)\tilde{\psi}(s) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \big|\tilde{\psi}(s)\big|^2\tilde{\psi}(s)\right] ds \right\|_{X^{1,(1/2)+}(I)}\right) \\
&\quad\leqslant c\big(\|\psi_{01}\|_{H^1} + \big\|(\tilde{\phi}_+ + \tilde{\phi}_-)\tilde{\psi}\big\|_{X^{1,-(1/2)+}(I)} + \big\||\tilde{\psi}|^2\tilde{\psi}\big\|_{X^{1,-(1/2)+}(I)}\big) \\
&\quad\leqslant c\big(\|\psi_{01}\|_{H^1} + \big\|(\tilde{\phi}_+ + \tilde{\phi}_-)\tilde{\psi}\big\|_{L_t^{\gamma'+}(I,H_x^{1,\rho'+})} + \big\||\tilde{\psi}|^2\tilde{\psi}\big\|_{L_t^{\gamma'+}(I,H_x^{1,\rho'+})}\big). \qquad (5.1)
\end{aligned}$$

Choosing $\gamma = \infty-$, $\rho = 2+$, we estimate by (4.4) and (4.5)

$$\begin{aligned}
\|\tilde{\phi}_\pm\tilde{\psi}\|_{H_x^{1,\rho'+}} &\leqslant c\big(\big\|(\nabla\tilde{\phi}_\pm)\tilde{\psi}\big\|_{L_x^{\rho'+}} + \big\|\tilde{\phi}_\pm(\nabla\tilde{\psi})\big\|_{L_x^{\rho'+}} + \|\tilde{\phi}_\pm\tilde{\psi}\|_{L_x^{\rho'+}}\big) \\
&\leqslant c\big(\|\nabla\tilde{\phi}_\pm\|_{L_x^2}\|\tilde{\psi}\|_{L_x^p} + \|\tilde{\phi}_\pm\|_{L_x^p}\|\nabla\tilde{\psi}\|_{L_x^2} + \|\tilde{\phi}_\pm\|_{L_x^p}\|\tilde{\psi}\|_{L_x^2}\big) \\
&\leqslant c\|A\tilde{\phi}_\pm\|_{L_x^2}\big(\|\nabla\tilde{\psi}\|_{L_x^2} + \|\tilde{\psi}\|_{L_x^2}\big) \leqslant cN^{4(1-s\wedge m)}, \\
\big\||\tilde{\psi}|^2\tilde{\psi}\big\|_{H_x^{1,\rho'+}} &\leqslant c\big(\big\||\tilde{\psi}|^2\tilde{\psi}\big\|_{L_x^{\rho'+}} + \big\|\nabla\tilde{\psi}\cdot|\tilde{\psi}|^2\big\|_{L_x^{\rho'+}}\big) \\
&\leqslant c\big(\|\tilde{\psi}\|_{L_x^{3\rho'+}}^3 + \|\nabla\tilde{\psi}\|\|\tilde{\psi}\|_{L_x^{2p}}^2\big) \\
&\leqslant c\left(\frac{3\rho'-2+}{2\rho'}\big[\|\nabla\tilde{\psi}\|^{\frac{3\rho'-2+}{\rho'}}\big]^{\frac{2\rho'}{3\rho'-2+}}\right. \\
&\qquad \left. + \frac{2-\rho'-}{2\rho'}\big[\|\tilde{\psi}\|^{\frac{2-}{\rho'}}\big]^{\frac{2\rho'}{2-\rho'-}} + \|\nabla\tilde{\psi}\|^{\frac{3p-2}{p}}\|\tilde{\psi}\|^{\frac{2}{p}}\right) \\
&\leqslant c\big(\|\nabla\tilde{\psi}\|^2 + \|\tilde{\psi}\|^{\frac{4-}{2-\rho'-}} + \|\nabla\tilde{\psi}\|^3 + \|\tilde{\psi}\|^3\big) \leqslant cN^{6(1-s\wedge m)},
\end{aligned}$$

where $\frac{1}{\rho'+} = \frac{1}{2} + \frac{1}{p}$. Thus

$$\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)} \leqslant c\big(N^{1-s} + N^{6(1-s\wedge m)}|I|^{1-}\big).$$

Assume now $|I| \leqslant N^{-5(1-\wedge m)-}$. Then we conclude

$$\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)} \leqslant cN^{1-s\wedge m}.$$

The same estimate also holds true, if we only assume $\|\psi_{01}\|_{H^1} \leqslant cN^{1-s\wedge m}$ (important remark for the iteration process described later).

Next we estimate $\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)}$. We again use (1.9) with $\gamma = \infty-$, $\rho = 2+$ and conclude as before

$$\begin{aligned}
\|\tilde{\phi}_\pm \tilde{\psi}\|_{L_t^{\gamma'+}(I,L_x^{\rho'+})} &\leqslant \|\tilde{\psi}\|_{L_t^\infty(I,L_x^2)} \|\tilde{\phi}_\pm\|_{L_t^\infty(I,L_x^p)} |I|^{1-} \\
&\leqslant cMN^{2(1-s\wedge m)} N^{-5(1-s\wedge m)} \leqslant c, \\
\||\tilde{\psi}|^2 \tilde{\psi}\|_{L_t^{\gamma'+}(I,L_x^{\rho'+})} &\leqslant \|\tilde{\psi}\|^3_{L_t^\infty(I,L_x^{3\rho'+})} |I|^{1-} \leqslant c\big(\|\nabla\tilde{\psi}\|^2 + \|\tilde{\psi}\|^{\frac{4-}{2-\rho'-}}\big)|I|^{1-} \\
&\leqslant c\big(N^{4(1-s\wedge m)} + M^{\frac{4-}{2-\rho'-}}\big) N^{-5(1-s\wedge m)} \leqslant c,
\end{aligned}$$

where $\frac{1}{\rho'+} = \frac{1}{2} + \frac{1}{p}$. Applying these estimates to (3.2) we get as before

$$\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)} \leqslant c\big(\|\psi_{01}\|_{L_x^2} + 1\big) \leqslant c$$

if $\|\psi_{01}\|_{L_x^2} \leqslant c$.

Last in order to estimate $\|\tilde{\phi}_\pm\|_{X_\pm^{1,(1/2)+}(I)}$, we use (1.13) to (3.2) and get

$$\begin{aligned}
&\|\tilde{\phi}_\pm\|_{X_\pm^{1,(1/2)+}(I)} \\
&\quad\leqslant c\left(\|\phi_{0\pm1}\|_{H_x^1} + \left\|\int_0^t U_\mp(t-s)A^{-1}\left[\big|\tilde{\psi}(s)\big|^2 - \frac{1}{8}\big(\tilde{\phi}_+(s) + \tilde{\phi}_-(s)\big)^3\right] ds\right\|_{X_\pm^{1,(1/2)+}(I)}\right) \\
&\quad\leqslant c\big(\|\phi_{0\pm1}\|_{H_x^1} + \||\tilde{\psi}|^2\|_{X_\pm^{0,-(1/2)+}(I)} + \|(\tilde{\phi}_+ + \tilde{\phi}_-)^3\|_{X_\pm^{0,-(1/2)+}(I)}\big) \\
&\quad\leqslant c\big(\|\phi_{0\pm1}\|_{H_x^1} + \||\tilde{\psi}|^2\|_{L_t^{1+}(I,L_x^2)} + \|(\tilde{\phi}_+ + \tilde{\phi}_-)^3\|_{L_t^{1+}(I,L_x^2)}\big).
\end{aligned}$$

We have

$$\begin{aligned}
\||\tilde{\psi}|^2\|_{L_t^{1+}(I,L_x^2)} &\leqslant c\|\tilde{\psi}\|_{L_t^\infty(I,L_x^2)} \|\nabla\tilde{\psi}\|_{L_t^\infty(I,L_x^2)} |I|^{1-} \\
&\leqslant cMN^{2(1-s\wedge m)} N^{-5(1-s\wedge m)} = cMN^{-3(1-s\wedge m)}, \\
\|(\tilde{\phi}_\pm)^3\|_{L_t^{1+}(I,L_x^2)} &\leqslant |I|^{1-} \|\tilde{\phi}_\pm\|^3_{L_t^\infty(I,L_x^6)} \leqslant cN^{6(1-s\wedge m)} N^{-5(1-s\wedge m)} = cN^{1-s\wedge m}, \\
\|(\tilde{\phi}_\pm)^2 \tilde{\phi}_\mp\|_{L_t^{1+}(I,L_x^2)} &\leqslant cN^{1-s\wedge m}.
\end{aligned}$$

Consequently

$$\|\tilde{\phi}_{\pm}\|_{X^{1,(1/2)+}_{\pm}(I)} \leqslant c\big(N^{1-m} + MN^{-3(1-s\wedge m)} + cN^{1-s\wedge m}\big).$$

Especially we conclude

$$\|\tilde{\phi}_{\pm}\|_{X^{1,(1/2)+}_{\pm}(I)} \leqslant cN^{1-s\wedge m}$$

if $\|\phi_{0\pm1}\|_{H^1_x} \leqslant cN^{1-s\wedge m}$, i.e. $\|\phi_{01}\|_{H^1_x} + \|\phi_{11}\|_{L^2_x} \leqslant cN^{1-s\wedge m}$.

We summarize the results obtained so far in the following.

Lemma 5.1. *If* $|I| \leqslant N^{-5(1-s\wedge m)-}$ *and*

$$\|\psi_{01}\|_{L^2_x} \leqslant c, \qquad \|\psi_{01}\|_{H^1_x} + \|\phi_{01}\|_{H^1_x} + \|\phi_{11}\|_{L^2_x} \leqslant cN^{1-s\wedge m}, \tag{5.2}$$

i.e.

$$\|\psi_{01}\|_{L^2_x} \leqslant c, \qquad \|\psi_{01}\|_{H^1_x} + \|\phi_{0\pm1}\|_{H^1_x} \leqslant cN^{1-s\wedge m}. \tag{5.3}$$

Then the following estimates hold:

$$\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)} \leqslant c, \qquad \|\tilde{\psi}\|_{X^{1,(1/2)+}(I)} \leqslant cN^{1-s\wedge m}, \qquad \|\tilde{\phi}_{\pm}\|_{X^{1,(1/2)+}_{\pm}(I)} \leqslant cN^{1-s\wedge m}.$$

Also the estimates (4.4)*,* (4.5) *hold under these assumptions.*

Remark. Here and in the sequel the constants denoted by c depend essentially only on $\bar{c}$ in (4.3) (and therefore on $E(\tilde{\psi}, \tilde{\phi}, \tilde{\phi}_t)$ and on M).

6. The part with rough data

In this section, we give a solution of our original system on I by constructing a solution about the system fulfilled by $(\hat{\psi}, \hat{\phi}_{\pm}) = (\psi - \tilde{\psi}, \phi_{\pm} - \tilde{\phi}_{\pm})$ with data $(\psi_{02}, \phi_{0\pm2})$ on the same interval I as the last section.

Let (ψ, ϕ_+, ϕ_-) be the solution of (3.1) with data $(\psi_0, \phi_{0+}, \phi_{0-})$ and $(\tilde{\psi}, \tilde{\phi}_+, \tilde{\phi}_-)$ be the solution with data $(\psi_{01}, \phi_{0+1}, \phi_{0-1})$.

Define $\hat{\psi} := \psi - \tilde{\psi}$, $\hat{\phi}_{\pm} := \phi_{\pm} - \tilde{\phi}_{\pm}$. Note that $\hat{\psi}$, $\hat{\phi}_{\pm}$ do not denote the Fourier transform of ψ and $\phi_{\pm}$ but the variations in this section. Then $(\hat{\psi}, \hat{\phi}_+, \hat{\phi}_-)$ satisfies

$$\begin{aligned}
i\hat{\psi}_t + \Delta\hat{\psi} &= i\psi_t + \Delta\psi - i\tilde{\psi}_t - \Delta\tilde{\psi} \\
&= -\frac{1}{2}\hat{\psi}(\hat{\phi}_+ + \hat{\phi}_-) - \frac{1}{2}\hat{\psi}\big(\tilde{\phi}_+ + \tilde{\phi}_-\big) - \frac{1}{2}\tilde{\psi}\big(\hat{\phi}_+ + \hat{\phi}_-\big) \\
&\quad + \tilde{\psi}\big(\tilde{\psi}\bar{\hat{\psi}} + 2\bar{\tilde{\psi}}\hat{\psi}\big) + \hat{\psi}\big(\bar{\tilde{\psi}}\hat{\psi} + 2\tilde{\psi}\bar{\hat{\psi}}\big) + |\hat{\psi}|^2\hat{\psi}
\end{aligned}$$

$$
\begin{aligned}
&=: F_1 + F_2 + F_3 + F_4 + F_5 + F_6 =: F,\\
i\hat{\phi}_{\pm t} \mp A\hat{\phi}_{\pm} &= i\phi_{\pm t} \mp A\phi_{\pm} - i\tilde{\phi}_{\pm t} \pm A\tilde{\phi}_{\pm}\\
&= \mp A^{-1}\big(|\hat{\psi}|^2\big) \mp A^{-1}\big(\bar{\hat{\psi}}\tilde{\psi}\big) \mp A^{-1}\big(\hat{\psi}\bar{\tilde{\psi}}\big)\\
&\quad \pm \frac{3}{8}A^{-1}\big[\big(\tilde{\phi}_+ + \tilde{\phi}_-\big)^2\big(\hat{\phi}_+ + \hat{\phi}_-\big)\big]\\
&\quad \pm \frac{3}{8}A^{-1}\big[\big(\tilde{\phi}_+ + \tilde{\phi}_-\big)\big(\hat{\phi}_+ + \hat{\phi}_-\big)^2\big] \pm \frac{1}{8}A^{-1}\big(\hat{\phi}_+ + \hat{\phi}_-\big)^3\\
&=: G_1 + G_2 + G_3 + G_4 + G_5 + G_6 =: G,\\
\hat{\psi}(0) &= \psi_0 - \psi_{01} = \psi_{02},\\
\hat{\phi}_{\pm}(0) &= \phi_{0\pm} - \phi_{0\pm 1} =: \phi_{0\pm 2}.
\end{aligned}
\tag{6.1}
$$

The corresponding system of integral equations are as follows:

$$
\hat{\psi}(t) = U(t)\psi_{02} - i\int_0^t U(t-s)F(s)\,ds =: U(t)\psi_{02} + w(t),
$$

$$
\hat{\phi}_{\pm}(t) = U_{\mp}(t)\phi_{0\pm 2} - i\int_0^t U_{\mp}(t-s)G(s)\,ds =: U_{\mp}(t)\phi_{0\pm 2} + z_{\pm}(t). \tag{6.2}
$$

Here we have

$$
\begin{aligned}
&\|\psi_{02}\|_{H_x^s} \leqslant c, && \|\psi_{02}\|_{L_x^2} \leqslant cN^{-s};\\
&\|\phi_{0\pm 2}\|_{H_x^m} \leqslant c, && \|\phi_{0\pm 2}\|_{L_x^2} \leqslant cN^{-m}.
\end{aligned}
\tag{6.3}
$$

We construct a solution of (6.2) in some time interval I using the fixed point argument. We define a mapping $S = (S_0, S_+, S_-)$ by

$$
\begin{aligned}
(S_0\hat{\psi})(t) &=: U(t)\psi_{02} + w(t),\\
(S_{\pm}\hat{\phi}_{\pm})(t) &=: U_{\mp}(t)\phi_{0\pm 2} + z_{\pm}(t).
\end{aligned}
$$

Proposition 6.1. *For* $\frac{1}{2} < s, m \leqslant 1$ *and* $\psi_{02} \in H_x^s(\mathbb{R}^2)$, $\phi_{0\pm 2} \in H_x^m(\mathbb{R}^2)$ *with* (6.3) *and* $\psi_{01}, \phi_{0\pm 1}$ *as in* (5.2), (5.3), *then the system of integral equations* (6.2) *has a unique solution* $\hat{\psi}, \hat{\phi}_{\pm} \in X^{s,\frac{1}{2}+}(I) \times X_{\pm}^{m,\frac{1}{2}+}(I)$ *in the same interval* I *of the preceding section with* $|I| \leqslant N^{-5(1-s\wedge m)-\delta}$ $(\delta > 0)$, *which satisfies*

$$
\begin{aligned}
&\|\hat{\psi}\|_{X^{0,(1/2)+}(I)} \leqslant cN^{-s} \leqslant cN^{-s\wedge m}, && \|\hat{\psi}\|_{X^{s,(1/2)+}(I)} \leqslant c,\\
&\|\hat{\phi}_{\pm}\|_{X_{\pm}^{0,(1/2)+}(I)} \leqslant cN^{-m} \leqslant cN^{-s\wedge m}, && \|\hat{\phi}_{\pm}\|_{X_{\pm}^{m,(1/2)+}(I)} \leqslant c.
\end{aligned}
$$

Proof. Define the closed set Z as

$$Z := \big\{ \|\hat{\psi}\|_{X^{s,(1/2)+}(I)} \leqslant c_0,\ \|\hat{\psi}\|_{X^{0,(1/2)+}(I)} \leqslant c_0 N^{-s},\\ \|\hat{\phi}_\pm\|_{X_\pm^{m,(1/2)+}(I)} \leqslant c_0,\ \|\hat{\phi}_\pm\|_{X_\pm^{0,(1/2)+}(I)} \leqslant c_0 N^{-m} \big\},$$

c_0 is chosen below.

Define the metric in the set Z as

$$\mathrm{d}\big((\psi_1,\phi_{1\pm}),(\psi_2,\phi_{2\pm})\big)\\ = \sup\big\{ \|\psi_1-\psi_2\|_{X^{s,(1/2)+}(I)}, \|\phi_{1+}-\phi_{2+}\|_{X_+^{m,(1/2)+}(I)}, \|\phi_{1-}-\phi_{2-}\|_{X_-^{m,(1/2)+}(I)} \big\}.$$

We use the fixed point argument in the set Z. Now take any $(\hat{\psi},\hat{\phi}_+,\hat{\phi}_-) \in Z$.

In order to estimate F and G, we use the following formulas which are given by interpolation and (1.4), formula (N1)

$$\|\hat{\psi}\|_{X^{\sigma,(1/2)+}(I)} \leqslant cN^{\sigma-s}, \qquad \|\hat{\psi}\|_{X^{\sigma,0}(I)} \leqslant c|I|^{1/2}N^{\sigma-s} \quad \text{for } 0\leqslant\sigma\leqslant s,\\ \|\hat{\phi}_\pm\|_{X_\pm^{\sigma,(1/2)+}(I)} \leqslant cN^{\sigma-m} \quad \text{for } 0\leqslant\sigma\leqslant m. \tag{N1}$$

In the same way, by Lemma 5.1 we have formula (N2)

$$\|\tilde{\psi}\|_{X^{\sigma,(1/2)+}(I)} \leqslant cN^{\sigma(1-s\wedge m)}, \qquad \|\tilde{\psi}\|_{X^{\sigma,0}(I)} \leqslant c|I|^{1/2}N^{\sigma(1-s\wedge m)} \quad \text{for } 0\leqslant\sigma\leqslant 1. \tag{N2}$$

We first estimate $\|S_0\hat{\psi}\|_{X^{s,(1/2)+}(I)}$. In order to estimate F_1, we use Lemma 2.6 with $\sigma = 0+$ and (N1) we have

$$\begin{aligned} &\|\hat{\psi}\hat{\phi}_\pm\|_{X^{s+,-(1/2)-}(I)}\\ &\quad\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{s-1+,(1/2)+}(I)} + \|\hat{\psi}\|_{X^{s,0}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\big)\\ &\quad\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)} + \|\hat{\psi}\|_{X^{s,0}(I)}\big)\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\\ &\quad\leqslant c\big(N^{-s} + N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big)N^{-m+} \leqslant cN^{-\gamma_1(s,m)}, \quad \gamma_1(s,m)>0. \end{aligned}\tag{6.4}$$

We estimate F_2 by Lemma 2.6 with $\sigma = \frac{1}{2}-$, (N1) and (N2) we have

$$\begin{aligned} &\|\hat{\psi}\tilde{\phi}_\pm\|_{X^{s+,-(1/2)-}(I)}\\ &\quad\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\tilde{\phi}_\pm\|_{X_\pm^{s-1+,(1/2)+}(I)} + \|\hat{\psi}\|_{X^{s-(1/2)+,0}(I)}\|\tilde{\phi}_\pm\|_{X_\pm^{1/2,(1/2)+}(I)}\big)\\ &\quad\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)} + \|\hat{\psi}\|_{X^{s-(1/2)+,0}(I)}\big)\|\tilde{\phi}_\pm\|_{X_\pm^{1,(1/2)+}(I)}\\ &\quad\leqslant c\big(N^{-s} + N^{-s(1-\frac{s-\frac{1}{2}}{s})+}N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big)N^{1-s\wedge m}\\ &\quad\leqslant cN^{-\gamma_2(s,m)}, \quad \gamma_2(s,m)>0. \end{aligned}\tag{6.5}$$

We estimate F_3 by Lemma 2.6 with $\sigma = 0+$, (N1) and (N2) we have

$$\begin{aligned}
\|\tilde{\psi}\hat{\phi}_\pm\|_{X^{s+,-(1/2)-}(I)} &\leqslant c\big(\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{s-1+,(1/2)+}(I)} + \|\tilde{\psi}\|_{X^{s,0}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\big) \\
&\leqslant c\big(\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)} + \|\tilde{\psi}\|_{X^{s,0}(I)}\big)\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)} \\
&\leqslant c\big(N^0 + N^{(s-\frac{5}{2})(1-s\wedge m)-\frac{\delta}{2}}\big)N^{-m+} \\
&\leqslant cN^{-\gamma_3(s,m)}, \quad \gamma_3(s,m) > 0. \qquad (6.6)
\end{aligned}$$

On the other hand, we get by Lemmas 3.1 and 5.1:

$$\begin{aligned}
\|\hat{\psi}\hat{\phi}_\pm\|_{X^{0,-(1/4)-\epsilon}(I)} &\leqslant c\|\hat{\psi}\|_{X^{0,(1/2)+\epsilon}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0,(1/2)+\epsilon}(I)} \leqslant cN^{-s}N^{-m} \leqslant cN^{-2s\wedge m}, \\
\|\hat{\psi}\tilde{\phi}_\pm\|_{X^{0,-(1/4)-\epsilon}(I)} &\leqslant c\|\hat{\psi}\|_{X^{0,(1/2)+\epsilon}(I)}\|\tilde{\phi}_\pm\|_{X_\pm^{0,(1/2)+\epsilon}(I)} \leqslant cN^{-s}N^{1-s\wedge m} \leqslant cN^{1-2s\wedge m}, \\
\|\tilde{\psi}\hat{\phi}_\pm\|_{X^{0,-(1/4)-\epsilon}(I)} &\leqslant c\|\tilde{\psi}\|_{X^{0,(1/2)+\epsilon}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0,(1/2)+\epsilon}(I)} \leqslant cN^{-m}. \qquad (6.7)
\end{aligned}$$

Now we interpolate between the above estimates and (6.4)–(6.6), and get

$$\begin{aligned}
\|\hat{\psi}\hat{\phi}_\pm\|_{X^{s,-(1/2)+}(I)} &\leqslant cN^{-\gamma_1(s,m)}, \\
\|\hat{\psi}\tilde{\phi}_\pm\|_{X^{s,-(1/2)+}(I)} &\leqslant cN^{-\gamma_2(s,m)}, \\
\|\tilde{\psi}\hat{\phi}_\pm\|_{X^{s,-(1/2)+}(I)} &\leqslant cN^{-\gamma_3(s,m)}. \qquad (6.8)
\end{aligned}$$

We estimate F_4, F_5, F_6 by (a2) with $\sigma_{ij} = 1/2$ for $i \neq j$, $\sigma_{ii} = 0$, $\alpha = 1/2$ and get

$$\begin{aligned}
\big\|(\tilde{\psi})^2\bar{\hat{\psi}}\big\|_{X^{s,-(1/2)+}(I)} &\leqslant c\big(\|\tilde{\psi}\|_{X^{s,(1/2)+}(I)}\|\tilde{\psi}\|_{X^{1/2,(1/2)+}(I)}\|\hat{\psi}\|_{X^{1/2,(1/2)+}(I)} \\
&\quad + \|\tilde{\psi}\|^2_{X^{1/2,(1/2)+}(I)}\|\hat{\psi}\|_{X^{s,(1/2)+}(I)}\big)|I|^{\frac{1}{2}-} \\
&\leqslant c\big(N^{(s+\frac{1}{2})(1-s\wedge m)-s(1-\frac{1}{2s})} + N^{1-s\wedge m}\big)N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}+} \\
&\leqslant cN^{-\gamma_4(s,m)}, \quad \gamma_4(s,m) > 0, \\
\big\||\tilde{\psi}|^2\hat{\psi}\big\|_{X^{s,-(1/2)+}(I)} &\leqslant cN^{-\gamma_4(s,m)}, \quad \gamma_4(s,m) > 0, \qquad (6.9) \\
\big\|\bar{\tilde{\psi}}(\hat{\psi})^2\big\|_{X^{s,-(1/2)+}(I)} &\leqslant c\big(\|\tilde{\psi}\|_{X^{1/2,(1/2)+}(I)}\|\hat{\psi}\|_{X^{s,(1/2)+}(I)}\|\hat{\psi}\|_{X^{1/2,(1/2)+}(I)} \\
&\quad + \|\tilde{\psi}\|_{X^{s,(1/2)+}(I)}\|\hat{\psi}\|^2_{X^{1/2,(1/2)+}(I)}\big)|I|^{\frac{1}{2}-} \\
&\leqslant c\big(N^{\frac{1}{2}(1-s\wedge m)}N^{\frac{1}{2}-s} + N^{s(1-s\wedge m)}N^{1-2s}\big)N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}+} \\
&\leqslant cN^{-\gamma_5(s,m)}, \quad \gamma_5(s,m) > 0, \\
\big\|\tilde{\psi}|\hat{\psi}|^2\big\|_{X^{s,-(1/2)+}(I)} &\leqslant cN^{-\gamma_5(s,m)}, \quad \gamma_5(s,m) > 0, \qquad (6.10)
\end{aligned}$$

$$\begin{aligned}\big\||\hat\psi|^2\hat\psi\big\|_{X^{s,-(1/2)+}(I)} &\leqslant c\|\hat\psi\|_{X^{s,(1/2)+}(I)}\|\hat\psi\|^2_{X^{1/2,(1/2)+}(I)}|I|^{\frac12-}\\ &\leqslant cN^{-s(2-\frac1s)}N^{-\frac52(1-s\wedge m)-\frac\delta2+}\\ &\leqslant cN^{-\gamma_6(s,m)},\quad \gamma_6(s,m)>0.\end{aligned}\tag{6.11}$$

Thus from (6.8)–(6.11), we have

$$\|F\|_{X^{s,-(1/2)+}(I)}\leqslant cN^{-\gamma(s,m)},\quad \gamma(s,m)>0.$$

Therefore, with $c_0\geqslant 2c\|\psi_{02}\|_{H^s(\mathbb{R}^2)}$, we have

$$\|S_0\hat\psi\|_{X^{s,(1/2)+}(I)}\leqslant c\|\psi_{02}\|_{H^s}+c\|F\|_{X^{s,-(1/2)+}(I)}\leqslant\frac{c_0}{2}+cN^{-\gamma(s,m)}\leqslant c_0,$$

if N is sufficiently large.

Next we estimate $\|S_0\hat\psi\|_{X^{0,(1/2)+}(I)}$. Using integral equations (6.2) we have to control $\|F\|_{X^{0,-(1/2)+}(I)}$.

We estimate F_1 by (1.9) with $\gamma'=\frac43$, $\rho'=\frac43$ and get

$$\begin{aligned}\|\hat\psi\hat\phi_\pm\|_{X^{0,-(1/2)+}(I)} &\leqslant c\|\hat\psi\hat\phi_\pm\|_{L_t^{(4/3)+}(I,L_x^{(4/3)+})}\leqslant c\|\hat\psi\|_{L_t^\infty(I,L_x^2)}\|\hat\phi_\pm\|_{L_t^\infty(I,L_x^{4+})}|I|^{\frac34-}\\ &\leqslant c\|\hat\psi\|_{L_t^\infty(I,L_x^2)}\|\hat\phi_\pm\|^{1-\frac{1}{2m}-}_{L_t^\infty(I,L_x^2)}\|\hat\phi_\pm\|^{\frac{1}{2m}+}_{L_t^\infty(I,H_x^m)}|I|^{\frac34-}\\ &\leqslant cN^{-s}N^{-m(1-\frac{1}{2m})+}N^{-\frac{15}{4}(1-s\wedge m)-\frac34\delta+}\leqslant cN^{-s-}.\end{aligned}\tag{6.12}$$

We estimate F_2 by (1.9) with $\gamma'=1+$, $\rho'=2-$ and Lemma 5.1 and get

$$\begin{aligned}\|\hat\psi\tilde\phi_\pm\|_{X^{0,-(1/2)+}(I)} &\leqslant \|\hat\psi\tilde\phi_\pm\|_{L_t^{1+}(I,L_x^{\rho'+}(\mathbb{R}^2))}\leqslant c\|\hat\psi\|_{L_t^\infty(I,L_x^2)}\|\tilde\phi_\pm\|_{L_t^\infty(I,L_x^p)}|I|^{1-}\\ &\leqslant cN^{-s}N^{1-s\wedge m}N^{-5(1-s\wedge m)-\delta+}\leqslant cN^{-s-},\end{aligned}\tag{6.13}$$

where $\frac12+\frac1p=\frac{1}{\rho'+}$.

We estimate F_3 by (1.9) with $\gamma'=\frac43$, $\rho'=\frac43$ again and Lemma 5.1 and get

$$\begin{aligned}\|\tilde\psi\hat\phi_\pm\|_{X^{0,-(1/2)+}(I)} &\leqslant c\|\tilde\psi\hat\phi_\pm\|_{L_t^{(4/3)+}(I,L_x^{(4/3)+})}\leqslant c\|\tilde\psi\|_{L_t^\infty(I,L_x^{4+})}\|\hat\phi_\pm\|_{L_t^\infty(I,L_x^2)}|I|^{\frac34-}\\ &\leqslant c\|\tilde\psi\|^{\frac12-}_{L_t^\infty(I,L_x^2)}\|\nabla\tilde\psi\|^{\frac12+}_{L_t^\infty(I,L_x^2)}\|\hat\phi_\pm\|_{L_t^\infty(I,L_x^2)}|I|^{\frac34-}\\ &\leqslant cN^{-m}N^{\frac12(1-s\wedge m)+}N^{-\frac{15}{4}(1-s\wedge m)-\frac34\delta+}\leqslant cN^{-s-}.\end{aligned}\tag{6.14}$$

Finally we estimate F_4, F_5, F_6 by (1.9) with $\gamma' = 1$, $\rho' = 2$ and Lemma 5.1 and get

$$\begin{aligned}
\big\|(\tilde\psi)^2\bar{\hat\psi}\big\|_{X^{0,-(1/2)+}(I)} &\leqslant c\big\|(\tilde\psi)^2\bar{\hat\psi}\big\|_{L_t^{1+}(I,L_x^2(\mathbb{R}^2))} \leqslant c\|\tilde\psi\|^2_{L_t^{12}(I,L_x^6(\mathbb{R}^2))}\|\hat\psi\|_{L_t^3(I,L_x^6(\mathbb{R}^2))}|I|^{\frac12-}\\
&\leqslant c\|\tilde\psi\|^2_{X^{1/2,(1/2)+}(I)}\|\hat\psi\|_{X^{0,(1/2)+}(I)}|I|^{\frac12-}\\
&\leqslant c\|\tilde\psi\|_{X^{0,(1/2)+}(I)}\|\tilde\psi\|_{X^{1,(1/2)+}(I)}\|\hat\psi\|_{X^{0,(1/2)+}(I)}|I|^{\frac12-}\\
&\leqslant cN^{-s}N^{1-s\wedge m}N^{-\frac52(1-s\wedge m)-\frac\delta2+} \leqslant cN^{-s-},\\
\big\||\tilde\psi|^2\hat\psi\big\|_{X^{0,-(1/2)+}(I)} &\leqslant cN^{-s-}, \qquad (6.15)\\
\big\|\tilde\psi(\hat\psi)^2\big\|_{X^{0,-(1/2)+}(I)} &\leqslant c\|\tilde\psi\|_{L_t^{12}(I,L_x^6(\mathbb{R}^2))}\|\hat\psi\|_{L_t^3(I,L_x^6(\mathbb{R}^2))}\|\hat\psi\|_{L_t^{12}(I,L_x^6(\mathbb{R}^2))}|I|^{\frac12-}\\
&\leqslant c\|\tilde\psi\|_{X^{1/2,(1/2)+}(I)}\|\hat\psi\|_{X^{0,(1/2)+}(I)}\|\hat\psi\|_{X^{1/2,(1/2)+}(I)}|I|^{\frac12-}\\
&\leqslant c\|\tilde\psi\|^{\frac12}_{X^{0,(1/2)+}(I)}\|\tilde\psi\|^{\frac12}_{X^{1,(1/2)+}(I)}\|\hat\psi\|^{2-\frac{1}{2s}}_{X^{0,(1/2)+}(I)}\|\hat\psi\|^{\frac{1}{2s}}_{X^{s,(1/2)+}(I)}|I|^{\frac12-}\\
&\leqslant cN^{\frac12(1-s\wedge m)}N^{-s(2-\frac{1}{2s})}N^{-\frac52(1-s\wedge m)-\frac\delta2+} \leqslant cN^{-s-},\\
\big\|\tilde\psi|\hat\psi|^2\big\|_{X^{0,-(1/2)+}(I)} &\leqslant cN^{-s-}, \qquad (6.16)\\
\big\||\hat\psi|^2\hat\psi\big\|_{X^{0,-(1/2)+}(I)} &\leqslant c\|\hat\psi\|_{L_t^3(I,L_x^6(\mathbb{R}^2))}\|\hat\psi\|^2_{L_t^{12}(I,L_x^6(\mathbb{R}^2))}|I|^{\frac12-}\\
&\leqslant c\|\hat\psi\|_{X^{0,(1/2)+}(I)}\|\hat\psi\|^2_{X^{1/2,(1/2)+}(I)}|I|^{\frac12-}\\
&\leqslant c\|\hat\psi\|^{3-\frac1s}_{X^{0,(1/2)+}(I)}\|\hat\psi\|^{\frac1s}_{X^{s,(1/2)+}(I)}|I|^{\frac12-}\\
&\leqslant cN^{-s}N^{-s(2-\frac1s)}N^{-\frac52(1-s\wedge m)-\frac\delta2+} \leqslant cN^{-s-}. \qquad (6.17)
\end{aligned}$$

From (6.2), (6.12)–(6.17), we conclude

$$\|S_0\hat\psi\|_{X^{0,(1/2)+}(I)} \leqslant c\|\psi_{02}\|_{L^2} + cN^{-s-} \leqslant cN^{-s} + cN^{-s-} \leqslant c_0N^{-s},$$

if $c_0 \geqslant 2c$ and N sufficiently large.

Next we estimate $\|S_\pm\hat\phi_\pm\|_{X_\pm^{m,(1/2)+}(I)}$. We need use Strichartz-type estimate in the case of $\theta = 0$ for the Klein–Gordon equation. Take

$$0 \leqslant \frac{2}{\gamma} = \frac12 - \frac1\rho \quad \text{and} \quad \mu = \frac32\left(\frac12 - \frac1\rho\right). \qquad (6.18)$$

In order to estimate G_2 we use (a3) with $\sigma_1 = 0$, $\sigma_2 = m-$, $\alpha = 1-$, we have

$$\begin{aligned}
\big\|\bar{\hat\psi}\tilde\psi\big\|_{X_\pm^{m-1,-(1/2)+}(I)} &\leqslant c|I|^{1-}\|\hat\psi\|_{X^{0,(1/2)+}(I)}\|\tilde\psi\|_{X^{m-,(1/2)+}(I)}\\
&\leqslant cN^{-s}N^{(1-s\wedge m)m-}N^{-5(1-s\wedge m)-\delta+} = cN^{-\gamma_2(s,m)}, \quad \gamma_2(s,m) > 0. \qquad (6.19)
\end{aligned}$$

We estimate G_3 in the same way.

Concerning G_1 we use (6.18) with $\mu = \frac{9}{8} - \frac{3}{4}m$, $\rho = \frac{2}{m-1/2}$, $\gamma = \frac{8}{3-2m}$ and get by the embeddings $H^{s+,p} \subset B^{s,p} \subset H^{s-,p}$ and the definition of the Besov spaces $B^{s,p}$ in [4,27]:

$$\begin{aligned}
&\big\| |\hat{\psi}|^2 \big\|_{H^{m-1+\mu-\epsilon,\rho'+}} \\
&\quad \leqslant c \big\| |\hat{\psi}|^2 \big\|_{B^{m-1+\mu,\rho'+}} \\
&\quad = c \left\{ \big\| |\hat{\psi}|^2 \big\|_{L^{\rho'+}} + \left(\int_0^\infty \Big[\tau^{-(m-1+\mu)} \sup_{|h|\leqslant\tau} \big\| |\hat{\psi}|^2(\cdot+h) - |\hat{\psi}|^2(\cdot) \big\|_{L^{\rho'+}} \Big]^2 \frac{d\tau}{\tau} \right)^{\frac{1}{2}} \right\} \\
&\quad \leqslant c \left\{ \|\hat{\psi}\|_{L^{\rho'\hat{p}+}} \|\hat{\psi}\|_{L^{\rho'\hat{q}+}} \right. \\
&\qquad \left. + \left(\int_0^\infty \Big[\tau^{-(m-1+\mu)} \sup_{|h|\leqslant\tau} \big\| \hat{\psi}(\cdot+h) - \hat{\psi}(\cdot) \big\|_{L^{\rho'\hat{q}+}} \Big]^2 \frac{d\tau}{\tau} \right)^{\frac{1}{2}} \|\hat{\psi}\|_{L^{\rho'\hat{p}+}} \right\} \\
&\quad \leqslant c \|\hat{\psi}\|_{L^{\rho'\hat{p}+}} \big(\|\hat{\psi}\|_{L^{\rho'\hat{q}+}} + \|\hat{\psi}\|_{B^{m-1+\mu,\rho'\hat{q}+}} \big) \leqslant c \|\hat{\psi}\|_{L^{\rho'\hat{p}+}} \|\hat{\psi}\|_{H^{m-1+\mu+\epsilon,\rho'\hat{q}+}}.
\end{aligned}$$

This implies

$$\big\| |\hat{\psi}|^2 \big\|_{L_t^{\gamma'+}(I, H_x^{m-1+\mu-,\rho'+})} \leqslant c \|\hat{\psi}\|_{L_t^\infty(I, L_x^{\rho'\hat{p}+})} \|\hat{\psi}\|_{L_t^\infty(I, H_x^{m-1-l+,\rho'\hat{q}+})} |I|^{\frac{1}{\gamma'}-}. \tag{6.20}$$

The Hölder exponents $\hat{p}$, $\hat{q}$ are chosen such that

$$H^s \subset L^{\rho'\hat{p}+} \cap H^{m-1+\mu+,\rho'\hat{q}+}.$$

This requires $\frac{1}{2} - \frac{s}{2} < \frac{1}{\rho'\hat{p}} < \frac{1}{2}$ and $\frac{1}{2} - \frac{s-(m-1+\mu)}{2} < \frac{1}{\rho'\hat{q}} < \frac{1}{2}$, which can be fulfilled if $s \wedge m > \frac{1}{2}$. Thus

$$\begin{aligned}
\big\| |\hat{\psi}|^2 \big\|_{L_t^{\gamma'+}(I, H_x^{m-1+\mu-,\rho'+})} &\leqslant c \|\hat{\psi}\|^2_{L_t^\infty(I, H_x^s)} |I|^{\frac{1}{\gamma'}-} \leqslant c N^{-5(1-s\wedge m)\frac{2m+5}{8} - \frac{2m+5}{8}\delta+} \\
&\leqslant c N^{-\gamma_1(s,m)}, \quad \gamma_1(s,m) > 0.
\end{aligned} \tag{6.21}$$

We estimate G_4, G_5, G_6 by Lemma 5.1 and get

$$\begin{aligned}
&\|(\tilde{\phi}_\pm)^2 \hat{\phi}_\pm\|_{X_\pm^{m-1,-(1/2)+}(I)} \leqslant c \big\|(\tilde{\phi}_\pm)^2 \hat{\phi}_\pm\big\|_{X_\pm^{0,-(1/2)+}(I)} \leqslant c \big\|(\tilde{\phi}_\pm)^2 \hat{\phi}_\pm\big\|_{L_t^{1+}(I, L_x^2(\mathbb{R}^2))} \\
&\quad \leqslant c \|\tilde{\phi}_\pm\|^2_{L_t^{12}(I, L_x^6(\mathbb{R}^2))} \|\hat{\phi}_\pm\|_{L_t^{12}(I, L_x^6(\mathbb{R}^2))} |I|^{\frac{3}{4}-} \\
&\quad \leqslant c \|\tilde{\phi}_\pm\|^2_{X_\pm^{2/3,(1/2)+}(I)} \|\hat{\phi}_\pm\|_{X_\pm^{2/3,(1/2)+}(I)} |I|^{\frac{3}{4}-}
\end{aligned}$$

$$\leqslant c\|\tilde{\phi}_\pm\|^{\frac{2}{3}}_{X^{0,(1/2)+}_\pm(I)}\|\tilde{\phi}_\pm\|^{\frac{4}{3}}_{X^{1,(1/2)+}_\pm(I)}\|\hat{\phi}_\pm\|^{1-\frac{2}{3m}}_{X^{0,(1/2)+}_\pm(I)}\|\hat{\phi}_\pm\|^{\frac{2}{3m}}_{X^{m,(1/2)+}_\pm(I)}|I|^{\frac{3}{4}-}$$

$$\leqslant cN^{\frac{4}{3}(1-s\wedge m)}N^{(1-\frac{2}{3m})(-m)}N^{-\frac{15}{4}(1-s\wedge m)-\frac{3}{4}\delta+}\leqslant cN^{-\gamma_4(s,m)},\quad \gamma_4(s,m)>0, \tag{6.22}$$

$$\big\|\tilde{\phi}_\pm(\hat{\phi}_\pm)^2\big\|_{X^{m-1,-(1/2)+}_\pm(I)}\leqslant c\|\tilde{\phi}_\pm\|_{X^{2/3,(1/2)+}_\pm(I)}\|\hat{\phi}_\pm\|^2_{X^{2/3,(1/2)+}_\pm(I)}|I|^{\frac{3}{4}-}$$

$$\leqslant c\|\tilde{\phi}_\pm\|^{\frac{1}{3}}_{X^{0,(1/2)+}_\pm(I)}\|\tilde{\phi}_\pm\|^{\frac{2}{3}}_{X^{1,(1/2)+}_\pm(I)}\|\hat{\phi}_\pm\|^{2-\frac{4}{3m}}_{X^{0,(1/2)+}_\pm(I)}\|\hat{\phi}_\pm\|^{\frac{4}{3m}}_{X^{m,(1/2)+}_\pm(I)}|I|^{\frac{3}{4}-}$$

$$\leqslant cN^{\frac{2}{3}(1-s\wedge m)}N^{(2-\frac{4}{3m})(-m)}N^{-\frac{15}{4}(1-s\wedge m)-\frac{3}{4}\delta+}$$

$$\leqslant cN^{-\gamma_5(s,m)},\quad \gamma_5(s,m)>0, \tag{6.23}$$

$$\big\|(\hat{\phi}_\pm)^3\big\|_{X^{m-1,-(1/2)+}_\pm(I)}$$

$$\leqslant c\|\hat{\phi}_\pm\|^3_{X^{2/3,(1/2)+}_\pm(I)}|I|^{\frac{3}{4}-}\leqslant c\|\hat{\phi}_\pm\|^{3-\frac{2}{m}}_{X^{0,(1/2)+}_\pm(I)}\|\hat{\phi}_\pm\|^{\frac{2}{m}}_{X^{m,(1/2)+}_\pm(I)}|I|^{\frac{3}{4}-}$$

$$\leqslant cN^{(3-\frac{2}{m})(-m)}N^{-\frac{15}{4}(1-s\wedge m)-\frac{3}{4}\delta+}\leqslant cN^{-\gamma_6(s,m)},\quad \gamma_6(s,m)>0. \tag{6.24}$$

From (6.2), (1.14), (6.19) and (6.21)–(6.24), we conclude

$$\|S_\pm\hat{\phi}_\pm\|_{X^{m,(1/2)+}}$$

$$\leqslant c\|\phi_{0\pm2}\|_{H^m}+c\|G\|_{X^{m,-(1/2)+}_\pm(I)}$$

$$\leqslant c\|\phi_{0\pm2}\|_{H^m}+c\big(\|\bar{\hat{\psi}}\tilde{\psi}\|_{L^{1+}_t(I,H^{m-1}_x)}+\|\hat{\psi}\bar{\tilde{\psi}}\|_{L^{1+}_t(I,H^{m-1}_x)}+\big\||\hat{\psi}|^2\big\|_{L^{\gamma'+}_t(I,H^{m-1+\mu-,\rho'+}_x)}$$

$$+\big\|(\tilde{\phi}_\pm)^2\hat{\phi}_\pm\big\|_{X^{m-1,-(1/2)+}_\pm(I)}+\big\|\tilde{\phi}_\pm(\hat{\phi}_\pm)^2\big\|_{X^{m-1,-(1/2)+}_\pm(I)}+\big\|(\hat{\phi}_\pm)^3\big\|_{X^{m-1,-(1/2)+}_\pm(I)}\big)$$

$$\leqslant c\|\phi_{0\pm2}\|_{H^m}+cN^{-\gamma(s,m)}\leqslant c_0,$$

provided $c_0\geqslant 2c\|\phi_{0\pm2}\|_{H^m}$ and N is sufficiently large.

Finally we estimate $\|S_\pm\hat{\phi}_\pm\|_{X^{0,(1/2)+}_\pm(I)}$ by Lemma 5.1 and Sobolev inequality

$$\begin{aligned}\big\||\hat{\psi}|^2\big\|_{L^{1+}_t(I,H^{-1}_x)}&\leqslant c\big\||\hat{\psi}|^2\big\|_{L^{1+}_t(I,L^1_x)}\leqslant c\|\hat{\psi}\|^2_{L^\infty_t(I,L^2_x)}|I|^{1-}\\&\leqslant cN^{-2s}N^{-5(1-s\wedge m)-\delta+}\leqslant cN^{-m-},\\ \|\bar{\hat{\psi}}\tilde{\psi}\|_{L^{1+}_t(I,H^{-1}_x)}&\leqslant c\|\bar{\hat{\psi}}\tilde{\psi}\|_{L^{1+}_t(I,L^1_x)}\leqslant c\|\hat{\psi}\|_{L^\infty_t(I,L^2_x)}\|\tilde{\psi}\|_{L^\infty_t(I,L^2_x)}|I|^{1-}\\&\leqslant cN^{-s}N^{-5(1-s\wedge m)-\delta+}\leqslant cN^{-m-}.\end{aligned} \tag{6.25}$$

The term $\|\hat{\psi}\bar{\tilde{\psi}}\|_{L^{1+}_t(I,H^{-1}_x)}$ is estimated in the same way.

$$\begin{aligned}\big\|(\tilde{\phi}_\pm)^2\hat{\phi}_\pm\big\|_{L^{1+}_t(I,H^{-1}_x(\mathbb{R}^2))}&\leqslant c\big\|(\tilde{\phi}_\pm)^2\hat{\phi}_\pm\big\|_{L^{1+}_t(I,L^1_x(\mathbb{R}^2))}\\&\leqslant c\|\tilde{\phi}_\pm\|^2_{L^\infty_t(I,L^4_x(\mathbb{R}^2))}\|\hat{\phi}_\pm\|_{L^\infty_t(I,L^2_x(\mathbb{R}^2))}|I|^{1-}\end{aligned}$$

$$\begin{aligned}
&\leqslant c\|\tilde{\phi}_\pm\|^2_{X_\pm^{1/2,(1/2)+}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0,(1/2)+}(I)}|I|^{1-}\\
&\leqslant cN^{2(1-s\wedge m)}N^{-m}N^{-5(1-s\wedge m)-\delta+}\leqslant cN^{-m-}, \qquad (6.26)\\
\big\|\tilde{\phi}_\pm(\hat{\phi}_\pm)^2\big\|_{L_t^{1+}(I,H_x^{-1}(\mathbb{R}^2))}&\leqslant c\|\tilde{\phi}_\pm\|_{X_\pm^{1/3,(1/2)+}(I)}\|\hat{\phi}_\pm\|^2_{X_\pm^{1/3,(1/2)+}(I)}|I|^{1-}\\
&\leqslant c\|\tilde{\phi}_\pm\|_{X_\pm^{1,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{2-\frac{2}{3m}}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{\frac{2}{3m}}_{X_\pm^{m,(1/2)+}(I)}|I|^{1-}\\
&\leqslant cN^{1-s\wedge m}N^{(2-\frac{2}{3m})(-m)}N^{-5(1-s\wedge m)-\delta+}\leqslant cN^{-m-}, \qquad (6.27)\\
\big\|(\hat{\phi}_\pm)^3\big\|_{L_t^{1+}(I,H_x^{-1}(\mathbb{R}^2))}&\leqslant c\|\hat{\phi}_\pm\|^3_{X_\pm^{1/3,(1/2)+}(I)}|I|^{1-}\\
&\leqslant c\|\hat{\phi}_\pm\|^{3-\frac{1}{m}}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{\frac{1}{m}}_{X_\pm^{m,(1/2)+}(I)}|I|^{1-}\\
&\leqslant cN^{(3-\frac{1}{m})(-m)}N^{-5(1-s\wedge m)-\delta+}\leqslant cN^{-m-}. \qquad (6.28)
\end{aligned}$$

From (6.2) and (6.25)–(6.28), we get

$$\|S_\pm\hat{\phi}_\pm\|_{X_\pm^{0,(1/2)+}(I)}\leqslant c\|\phi_{0\pm2}\|_{L^2}+cN^{-m-}\leqslant c_0N^{-m},$$

if $c_0\geqslant 2c$ and N sufficiently large.

Summarizing, we have shown that S maps Z into itself. The contraction property uses exactly the same type of estimates and is therefore omitted.

Proposition 6.2. *Under the assumptions of Proposition* 6.1*, then the following estimate holds with* $q=3/\gamma$*,* $r=6/(3-4\gamma)$*:*

$$\|\hat{\phi}_\pm\|_{L_t^\infty(I,\dot{H}_x^\gamma(\mathbb{R}^2))}+\|\hat{\phi}_\pm\|_{L_t^q(I,L_x^r(\mathbb{R}^2))}\leqslant cN^{\gamma-m},\quad \textit{if } 0\leqslant\gamma\leqslant\frac{3}{4}\wedge m.$$

Proof. Consider the case $\gamma=3/4\leqslant m$. Then $q=4$, $r=\infty$. First, since

$$\big\|U_\pm(t)\phi\big\|_{L_t^4(I,L_x^\infty)}\leqslant\|\phi\|_{H_x^{3/4}},$$

we have

$$\big\|U_\pm(t)\phi_{0\pm2}\big\|_{L_t^4(I,L_x^\infty)}\leqslant cN^{3/4-m}.$$

Secondly,

$$\begin{aligned}
\bigg\|\int_0^t U_\mp(t-s)G(s)\,ds\bigg\|_{L_t^4(I,L_x^\infty)}&\leqslant c\bigg\|\int_0^t U_\mp(t-s)G(s)\,ds\bigg\|_{X_\pm^{3/4,(1/2)+}(I)}\\
&\leqslant c\big\|G(t)\big\|_{X_\pm^{3/4,-(1/2)+}(I)}
\end{aligned}$$

$$\begin{aligned}&\leqslant c\big\|G(t)\big\|_{L_t^{1+}(I,H_x^{3/4})}\leqslant c\big\|AG(t)\big\|_{L_t^{1+}(I,H_x^{-1/4})}\\&\leqslant c\big\|AG(t)\big\|_{L_t^{1+}(I,L_x^{8/5})}.\end{aligned}$$

On the other hand, since $\|f\|_{L_x^8(\mathbb{R}^2)}\leqslant c\|f\|_{H_x^{3/4}(\mathbb{R}^2)}$, we get

$$\begin{aligned}\|fg\|_{L_t^{1+}(I,L_x^{8/5})}=\|f\bar g\|_{L_t^{1+}(I,L_x^{8/5})}=\|\bar f g\|_{L_t^{1+}(I,L_x^{8/5})}&=\|\bar f\bar g\|_{L_t^{1+}(I,L_x^{8/5})}\\&\leqslant c|I|^{1-}\|f\|_{L_t^\infty(I,L_x^2)}\|g\|_{L_t^\infty(I,H_x^{3/4})}\\&\leqslant c|I|^{1-}\|f\|_{X^{0,1/2+}(I)}\|g\|_{X^{3/4,1/2+}(I)}.\end{aligned}$$

Therefore, we get

$$\big\|\bar{\hat\psi}\tilde\psi\big\|_{L_t^{1+}(I,L_x^{8/5})}\leqslant c|I|^{1-}\|\hat\psi\|_{X^{0,(1/2)+}(I)}\|\tilde\psi\|_{X^{3/4,1/2+}(I)}\leqslant cN^\beta,$$

where $\beta=(\frac34-5)(1-s\wedge m)-s\leqslant\frac34-m$.

$\||\hat\psi|^2\|_{L_t^{1+}(I,L_x^{8/5})}$, $\|\hat\psi\bar{\tilde\psi}\|_{L_t^{1+}(I,L_x^{8/5})}$ are estimated in the same way.

By $\|v_1v_2v_3\|_{L_t^{1+}(I,L_x^{8/5})}\leqslant c|I|^{1-}\|v_1\|_{X_\pm^{1/2,(1/2)+}(I)}\|v_2\|_{X_\pm^{1/2,1/2+}(I)}\|v_3\|_{X_\pm^{3/4,1/2+}(I)}$ we see that

$$\big\|(\tilde\phi_\pm)^2\hat\phi_\pm\big\|_{L_t^{1+}(I,L_x^{8/5})}\leqslant cN^{3/4-m}.$$

We see also that $\|\tilde\phi_\pm(\hat\phi_\pm)^2\|_{L_t^{1+}(I,L_x^{8/5})}$, $\|(\hat\phi_\pm)^3\|_{L_t^{1+}(I,L_x^{8/5})}$ are estimated by $cN^{3/4-m}$.

Thus we see that $\|AG(t)\|_{L_t^{1+}(I,L_x^{8/5})}\leqslant cN^{3/4-m}$. Hence we have Proposition 6.2 for the case $\gamma=3/4\leqslant m$.

For the case where $\gamma=m\leqslant\frac34$, we get the desired result by Proposition 6.1 and (1.11).

Since the inequality for the case $\gamma=0$ is trivial, by interpolation we get the inequality for the case $0<\gamma<\frac34\wedge m$.

This completes the proof. □

The next estimates shown that the nonlinear part w in the first integral equation of (6.2) behaves better than the corresponding linear part.

Proposition 6.3. *Under the assumptions of Proposition* 6.1*, then the following estimates hold*:

$$\begin{aligned}&\|w\|_{L_t^\infty(I,L_x^2)}\leqslant\|w\|_{X^{0,(1/2)+}(I)}\leqslant cN^{-\frac14-\frac34 s\wedge m},\\&\|w\|_{L_t^\infty(I,H_x^1)}\leqslant\|w\|_{X^{1,(1/2)+}(I)}\leqslant cN^{-1+\frac12 s\wedge m},\quad \textit{if } s\wedge m>\frac45.\end{aligned}$$

Proof. By (1.6), (6.7), (6.15)–(6.17) and Lemma 5.1, we have

$$\begin{aligned}
\|w\|_{X^{0,(1/2)+}(I)} &\leqslant c\big(\|\hat{\phi}_\pm\hat{\psi}\|_{X^{0,-(1/4)-}(I)} + \|\tilde{\phi}_\pm\hat{\psi}\|_{X^{0,-(1/4)-}(I)} + \|\hat{\phi}_\pm\tilde{\psi}\|_{X^{0,-(1/4)-}(I)}\big)|I|^{\frac{1}{4}-} \\
&\quad + \big\|(\tilde{\psi})^2\bar{\hat{\psi}}\big\|_{X^{0,-(1/2)+}(I)} + \big\|\tilde{\psi}(\hat{\psi})^2\big\|_{X^{0,-(1/2)+}(I)} + \big\||\hat{\psi}|^2\hat{\psi}\big\|_{X^{0,-(1/2)+}(I)} \\
&\quad + \big\||\tilde{\psi}|^2\hat{\psi}\big\|_{X^{0,-(1/2)+}(I)} + \big\|\tilde{\psi}|\hat{\psi}|^2\big\|_{X^{0,-(1/2)+}(I)} \\
&\leqslant c\big(N^{-s\wedge m}N^{-s\wedge m} + N^{1-s\wedge m}N^{-s\wedge m} + N^{-s\wedge m}\big)N^{-\frac{5}{4}(1-s\wedge m)-\frac{1}{4}\delta+} \\
&\quad + c\big(N^{1-s\wedge m}N^{-s\wedge m} + N^{\frac{1}{2}(1-s\wedge m)}N^{-s\wedge m(2-\frac{1}{2s})} + N^{-s\wedge m(3-\frac{1}{s})}\big) \\
&\quad \times N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}+} \\
&\leqslant cN^{-\frac{1}{4}-\frac{3}{4}s\wedge m}.
\end{aligned} \tag{6.29}$$

Concerning $\|w\|_{X^{1,(1/2)+}(I)}$, we have

$$\begin{aligned}
\|w\|_{X^{1,(1/2)+}(I)} &\leqslant c\big(\|\hat{\phi}_\pm\hat{\psi}\|_{X^{1,-(1/2)+}(I)} + \|\tilde{\phi}_\pm\hat{\psi}\|_{X^{1,-(1/2)+}(I)} + \|\hat{\phi}_\pm\tilde{\psi}\|_{X^{1,-(1/2)+}(I)} \\
&\quad + \big\|(\tilde{\psi})^2\bar{\hat{\psi}}\big\|_{X^{1,-(1/2)+}(I)} + \big\|\tilde{\psi}(\hat{\psi})^2\big\|_{X^{1,-(1/2)+}(I)} + \big\||\hat{\psi}|^2\hat{\psi}\big\|_{X^{1,-(1/2)+}(I)} \\
&\quad + \big\||\tilde{\psi}|^2\hat{\psi}\big\|_{X^{1,-(1/2)+}(I)} + \big\|\tilde{\psi}|\hat{\psi}|^2\big\|_{X^{1,-(1/2)+}(I)}\big).
\end{aligned} \tag{6.30}$$

We consider separately the cases $s \leqslant m$ and $m \leqslant s$.

Case 1: $s \leqslant m$. By Lemma 2.6 with $s = 1+$ and $\sigma = m-$ and interpolation

$$\begin{aligned}
\|\hat{\phi}_\pm\hat{\psi}\|_{X^{1+,-(1/2)-}(I)} &\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)} + \|\hat{\psi}\|_{X^{1-m+,0}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{m,(1/2)+}(I)}\big) \\
&\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{1-}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{0+}_{X_\pm^{m,(1/2)+}(I)} \\
&\quad + \|\hat{\psi}\|^{1-\frac{1-m}{s}-}_{X^{0,(1/2)+}(I)}\|\hat{\psi}\|^{\frac{1-m}{s}+}_{X^{s,(1/2)+}(I)}|I|^{\frac{1}{2}}\|\hat{\phi}_\pm\|_{X_\pm^{m,(1/2)+}(I)}\big) \\
&\leqslant c\big(N^{-s}N^{-s\wedge m} + N^{-s(1-\frac{1-m}{s})+}N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big) \\
&\leqslant N^{-1+\frac{1}{2}s}.
\end{aligned} \tag{6.31}$$

Next by Lemma 2.6 with $s = 1+$ and $\sigma = \frac{1}{2}-$ we get by interpolation

$$\begin{aligned}
&\|\tilde{\phi}_\pm\hat{\psi}\|_{X^{1+,-(1/2)-}(I)} \\
&\quad \leqslant c\big(\|\tilde{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0,(1/2)+}(I)} + \|\tilde{\phi}_\pm\|_{X_\pm^{1/2,(1/2)+}(I)}\|\hat{\psi}\|_{X^{(1/2)+,0}(I)}\big) \\
&\quad \leqslant c\big(\|\tilde{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0,(1/2)+}(I)} \\
&\qquad + \|\tilde{\phi}_\pm\|_{X_\pm^{1/2,(1/2)+}(I)}\|\hat{\psi}\|^{1-\frac{1}{2s}-}_{X^{0,(1/2)+}(I)}\|\hat{\psi}\|^{\frac{1}{2s}+}_{X^{s,(1/2)+}(I)}|I|^{\frac{1}{2}}\big)
\end{aligned}$$

$$\leqslant c\big(N^{1-s\wedge m}N^{-s}+N^{1-s\wedge m}N^{-s(1-\frac{1}{2s})+}N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big)$$
$$\leqslant c\big(N^{1-2s}+N^{1-s-s+\frac{1}{2}-\frac{5}{2}+\frac{5}{2}s}\big)\leqslant cN^{-1+\frac{1}{2}s}. \tag{6.32}$$

Moreover from Lemma 2.6 with $s=1+$ and $\sigma=0+$, Lemma 5.1 and interpolation

$$\begin{aligned}
&\|\hat{\phi}_\pm\tilde{\psi}\|_{X^{1+,-(1/2)-}(I)}\\
&\quad\leqslant c\big(\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}+\|\tilde{\psi}\|_{X^{1,0}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\big)\\
&\quad=c\big(\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)}+\|\tilde{\psi}\|_{X^{1,0}(I)}\big)\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)}\\
&\quad\leqslant c\big(\|\tilde{\psi}\|_{X^{0,(1/2)+}(I)}+\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)}|I|^{\frac{1}{2}}\big)\|\hat{\phi}_\pm\|^{1-}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{0+}_{X_\pm^{m,(1/2)+}(I)}\\
&\quad\leqslant c\big(1+N^{1-s\wedge m}N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big)N^{-s\wedge m+}\\
&\quad\leqslant c\big(N^{-s\wedge m+}+N^{-\frac{3}{2}+\frac{1}{2}s\wedge m}\big)\leqslant cN^{-1+\frac{1}{2}s}.
\end{aligned} \tag{6.33}$$

From Lemmas 2.2 and 5.1, we have

$$\begin{aligned}
\big\|(\tilde{\psi})^2\bar{\hat{\psi}}\big\|_{X^{1,-(1/2)+}(I)}&\leqslant\sup_{\|w\|_{X^{0,(1/2)-}}\leqslant1}\big|\big\langle w,D_x\big[(\tilde{\psi})^2\bar{\hat{\psi}}\big]\big\rangle\big|\\
&\leqslant c\sup_{\|w\|_{X^{0,(1/2)-}}\leqslant1}\big(\big|\big\langle w,D_x\tilde{\psi}\cdot\tilde{\psi}\bar{\hat{\psi}}\big\rangle\big|+\big|\big\langle w,\tilde{\psi}\cdot D_x(\tilde{\psi}\bar{\hat{\psi}})\big\rangle\big|\big)\\
&\leqslant c\sup_{\|w\|_{X^{0,(1/2)-}}\leqslant1}\big(\|wD_x\bar{\tilde{\psi}}\|_{L^{2-}_{tx}}\|\tilde{\psi}\bar{\hat{\psi}}\|_{L^{2+}_{tx}}\\
&\qquad+\big\|D_x^{\frac{1}{2}-}(w\bar{\tilde{\psi}})\big\|_{L^2_{tx}}\big\|D_x^{\frac{1}{2}+}(\tilde{\psi}\bar{\hat{\psi}})\big\|_{L^2_{tx}}\big)\\
&\leqslant c\sup_{\|w\|_{X^{0,(1/2)-}}}\big(\|w\|_{L^{4-}_{tx}}\|D_x\tilde{\psi}\|_{L^4_{tx}}\|\tilde{\psi}\|_{L^{4+}_{tx}}\|\hat{\psi}\|_{L^4_{tx}}\\
&\qquad+\big\|D_x^{\frac{1}{2}-}(w\bar{\tilde{\psi}})\big\|_{L^2_{tx}}\big\|D_x^{\frac{1}{2}+}(\tilde{\psi}\bar{\hat{\psi}})\big\|_{L^2_{tx}}\big)\\
&\leqslant c\big(\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)}\|\tilde{\psi}\|_{X^{0+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\\
&\qquad+\|\tilde{\psi}\|_{X^{(1/2)+,(1/2)+}(I)}\|\tilde{\psi}\|_{X^{(1/2)+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0+,(1/2)+}(I)}\big)\\
&\leqslant c\big(\|\tilde{\psi}\|^{1-}_{X^{0,(1/2)+}(I)}\|\tilde{\psi}\|^{1+}_{X^{1,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\\
&\qquad+\|\tilde{\psi}\|^{1-}_{X^{0,(1/2)+}(I)}\|\tilde{\psi}\|^{1+}_{X^{1,(1/2)+}(I)}\|\hat{\psi}\|^{1-}_{X^{0,(1/2)+}(I)}\|\hat{\psi}\|^{0+}_{X^{s,(1/2)+}(I)}\big)\\
&\leqslant c\big(N^{1-s\wedge m+}N^{-s\wedge m}+N^{1-s\wedge m+}N^{-s\wedge m+}\big)\leqslant cN^{1-2s+},\\
\big\||\tilde{\psi}|^2\hat{\psi}\big\|_{X^{1,-(1/2)+}(I)}&\leqslant cN^{1-2s+},
\end{aligned} \tag{6.34}$$

$$
\begin{aligned}
\big\|\tilde{\psi}(\hat{\psi})^2\big\|_{X^{1,-(1/2)+}(I)} &\leqslant c \sup_{\|w\|_{X^{0,(1/2)-}}\leqslant 1} \big(\big|\big\langle w, D_x\bar{\tilde{\psi}}\cdot(\hat{\psi})^2\big\rangle\big| + \big|\big\langle w, \bar{\tilde{\psi}}\cdot D_x(\hat{\psi})^2\big\rangle\big|\big) \\
&\leqslant c \sup_{\|w\|_{X^{0,(1/2)-}}\leqslant 1} \big(\|wD_x\tilde{\psi}\|_{L^{2-}_{tx}}\|(\hat{\psi})^2\|_{L^{2+}_{tx}} \\
&\quad + \big\|D_x^{\frac{1}{2}-}(w\tilde{\psi})\big\|_{L^2_{tx}}\big\|D_x^{\frac{1}{2}+}(\hat{\psi})^2\big\|_{L^2_{tx}}\big) \\
&\leqslant c \sup_{\|w\|_{X^{0,(1/2)-}}} \big(\|w\|_{L^{4-}_{tx}}\|D_x\tilde{\psi}\|_{L^4_{tx}}\|\hat{\psi}\|^2_{L^{4+}_{tx}} \\
&\quad + \big\|D_x^{\frac{1}{2}-}(w\tilde{\psi})\big\|_{L^2_{tx}}\big\|D_x^{\frac{1}{2}+}(\hat{\psi})^2\big\|_{L^2_{tx}}\big) \\
&\leqslant c\big(\|\tilde{\psi}\|_{X^{1,(1/2)+}(I)}\|\hat{\psi}\|^2_{X^{0+,(1/2)+}(I)} \\
&\quad + \|\tilde{\psi}\|_{X^{(1/2)+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{(1/2)+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0+,(1/2)+}(I)}\big) \\
&\leqslant c\big(N^{1-s\wedge m+}N^{-2s\wedge m} + N^{\frac{1}{2}(1-s\wedge m)+}N^{-s\wedge m(2-\frac{1}{2s})+}\big) \\
&\leqslant cN^{1-\frac{5}{2}s+}, \\
\big\|\tilde{\psi}|\hat{\psi}|^2\big\|_{X^{1,-(1/2)+}(I)} &\leqslant cN^{1-\frac{5}{2}s+}, && (6.35) \\
\big\||\hat{\psi}|^2\hat{\psi}\big\|_{X^{1,-(1/2)+}(I)} &\leqslant c \sup_{\|w\|_{X^{0,(1/2)-}}\leqslant 1} \big|\big\langle w, D_x\hat{\psi}\cdot|\hat{\psi}|^2\big\rangle\big| \\
&\leqslant c \sup_{\|w\|_{X^{0,(1/2)-}}\leqslant 1} \big|\big\langle w\hat{\psi}, D_x(\hat{\psi})^2\big\rangle\big| \\
&\leqslant c \sup_{\|w\|_{X^{0,(1/2)-}}\leqslant 1} \big\|D_x^{\frac{1}{2}-}(w\hat{\psi})\big\|_{L^2_{tx}}\big\|D_x^{\frac{1}{2}+}(\hat{\psi})^2\big\|_{L^2_{tx}} \\
&\leqslant c\|\hat{\psi}\|_{X^{(1/2)+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{(1/2)+,(1/2)+}(I)}\|\hat{\psi}\|_{X^{0+,(1/2)+}(I)} \\
&\leqslant c\|\hat{\psi}\|^{3-\frac{1}{s}-}_{X^{0,(1/2)+}(I)}\|\hat{\psi}\|^{\frac{1}{s}+}_{X^{s,(1/2)+}(I)} \\
&\leqslant cN^{-s\wedge m(3-\frac{1}{s})+} \leqslant cN^{1-3s+}. && (6.36)
\end{aligned}
$$

Case 2: $m \leqslant s$. By Lemma 2.6 with $s = 1+$ and $\sigma = 1 - s+$ and interpolation,

$$
\begin{aligned}
\|\hat{\phi}_\pm\hat{\psi}\|_{X^{1+,-(1/2)-}(I)} &\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{0+,(1/2)+}(I)} + \|\hat{\psi}\|_{X^{s,0}(I)}\|\hat{\phi}_\pm\|_{X_\pm^{1-s+,(1/2)+}(I)}\big) \\
&\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{1-}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{0+}_{X_\pm^{m,(1/2)+}(I)} \\
&\quad + \|\hat{\psi}\|_{X^{s,(1/2)+}(I)}|I|^{\frac{1}{2}}\|\hat{\phi}_\pm\|_{X_\pm^{1-s+,(1/2)+}(I)}\big) \\
&\leqslant c\big(\|\hat{\psi}\|_{X^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{1-}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{0+}_{X_\pm^{m,(1/2)+}(I)} \\
&\quad + \|\hat{\psi}\|_{X^{s,(1/2)+}(I)}|I|^{\frac{1}{2}}\|\hat{\phi}_\pm\|^{1-\frac{1-s}{m}-}_{X_\pm^{0,(1/2)+}(I)}\|\hat{\phi}_\pm\|^{\frac{1-s}{m}+}_{X_\pm^{m,(1/2)+}(I)}\big)
\end{aligned}
$$

$$\leqslant c\big(N^{-s\wedge m}N^{-m+} + N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}N^{-m(1-\frac{1-s}{m})+}\big)$$
$$\leqslant c(N^{-2m+} + N^{-\frac{3}{2}-s+\frac{3}{2}m-\frac{\delta}{2}+}) \leqslant cN^{-1+\frac{1}{2}m}.$$

From (6.32) we conclude

$$\|\tilde{\phi}_\pm\hat{\psi}\|_{X^{1+,-(1/2)-}(I)} \leqslant c\big(N^{1-s\wedge m}N^{-s\wedge m} + N^{1-s\wedge m}N^{-s\wedge m(1-\frac{1}{2s})+}N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big)$$
$$\leqslant c\big(N^{1-2m} + N^{-\frac{3}{2}+m(\frac{1}{2}+\frac{1}{2s})}\big) \leqslant cN^{-1+\frac{1}{2}m}.$$

From (6.33)

$$\|\hat{\phi}_\pm\tilde{\psi}\|_{X^{1+,-(1/2)-}(I)} \leqslant c\big(1 + N^{1-s\wedge m}N^{-\frac{5}{2}(1-s\wedge m)-\frac{\delta}{2}}\big)N^{-m+}$$
$$\leqslant c\big(N^{-m+} + N^{-\frac{3}{2}+\frac{1}{2}m}\big) \leqslant cN^{-1+\frac{1}{2}m}.$$

From (6.34)

$$\big\|(\tilde{\psi})^2\bar{\hat{\psi}}\big\|_{X^{1,-(1/2)+}(I)} \leqslant c\big(N^{1-m+}N^{-m} + N^{1-m+}N^{-m+}\big) \leqslant cN^{1-2m+},$$
$$\big\||\tilde{\psi}|^2\hat{\psi}\big\|_{X^{1,-(1/2)+}(I)} \leqslant cN^{1-2m+}.$$

From (6.35)

$$\big\|\tilde{\psi}(\hat{\psi})^2\big\|_{X^{1,-(1/2)+}(I)} \leqslant c\big(N^{1-m+}N^{-2m} + N^{\frac{1}{2}(1-m)+}N^{-m(2-\frac{1}{2s})+}\big) \leqslant cN^{1-\frac{5}{2}m+},$$
$$\big\|\tilde{\psi}|\hat{\psi}|^2\big\|_{X^{1,-(1/2)+}(I)} \leqslant cN^{1-\frac{5}{2}m+}.$$

From (6.36)

$$\big\||\hat{\psi}|^2\hat{\psi}\big\|_{X^{1,-(1/2)+}(I)} \leqslant cN^{-m(3-\frac{1}{s})+} \leqslant cN^{1-3m+}.$$

Summarizing we get

$$\begin{aligned}\|w\|_{X^{1,(1/2)+}(I)} &\leqslant c\|F\|_{X^{1,-(1/2)+}(I)}\\ &\leqslant c\big(\|F_1\|_{X^{1+,-(1/2)+}(I)} + \|F_2\|_{X^{1+,-(1/2)+}(I)} + \|F_3\|_{X^{1+,-(1/2)+}(I)}\\ &\quad + \|F_4\|_{X^{1,-(1/2)+}(I)} + \|F_5\|_{X^{1,-(1/2)+}(I)} + \|F_6\|_{X^{1,-(1/2)+}(I)}\big)\\ &\leqslant cN^{1-\frac{3}{2}(s\wedge m)}.\end{aligned}$$

We complete the proof. □

The next result show that the nonlinear part $z_\pm$ in the second integral equation of (6.2) also behaves better than the corresponding linear part.

Proposition 6.4. *Under the assumptions of Proposition* 6.1*, the following estimates hold*:

$$\|z_\pm\|_{L^\infty(I,L^2)} \leqslant \|z_\pm\|_{X_\pm^{0,(1/2)+}(I)} \leqslant cN^{-m} \leqslant cN^{-s\wedge m},$$

$$\|z_\pm\|_{L^\infty(I,H^1)} + \|z_{\pm t}\|_{L^\infty(I,L^2)} \leqslant cN^{-3+2s\wedge m}, \quad \text{if } s\wedge m > \frac{4}{5}.$$

Proof. The estimate $\|z_\pm\|_{X_\pm^{0,(1/2)+}(I)}$ is already proven in Proposition 6.1. From (6.2), we have

$$\begin{aligned}
&\|z_\pm\|_{L^\infty(I,H^1)} \\
&\quad \leqslant c\big(\big\||\hat\psi|^2\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} + \|\bar{\tilde\psi}\tilde\psi\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} + \|\hat\psi\bar{\tilde\psi}\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} \\
&\qquad + \big\|(\tilde\phi_\pm)^2\hat\phi_\pm\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} + \big\|\tilde\phi_\pm(\hat\phi_\pm)^2\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} + \big\|(\hat\phi_\pm)^3\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))}\big).
\end{aligned}$$

It suffices to estimate the first, second, forth and sixth terms. From Lemma 5.1, Propositions 6.1 and 6.2, we have

$$\begin{aligned}
\big\||\hat\psi|^2\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} &\leqslant |I|^{\frac12}\|\hat\psi\|^2_{L_t^4(I,L_x^4(\mathbb{R}^2))} \leqslant c|I|^{\frac12}\|\hat\psi\|^2_{X^{0,(1/2)+}(I)} \\
&\leqslant cN^{-\frac52(1-s\wedge m)}N^{-2s\wedge m} \leqslant cN^{-\frac52+\frac12 s\wedge m}, \\
\|\bar{\tilde\psi}\tilde\psi\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} &\leqslant |I|\|\tilde\psi\|_{L_t^\infty(I,L_x^{\infty-}(\mathbb{R}^2))}\|\hat\psi\|_{L_t^\infty(I,L_x^{2+}(\mathbb{R}^2))} \\
&\leqslant c|I|\|\tilde\psi\|_{L_t^\infty(I,H_x^1(\mathbb{R}^2))}\|\hat\psi\|^{1-}_{L_t^\infty(I,L_x^2(\mathbb{R}^2))}\|\hat\psi\|^{0+}_{L_t^\infty(I,H_x^s(\mathbb{R}^2))} \\
&\leqslant cN^{-5(1-s\wedge m)-\delta}N^{1-s\wedge m}N^{-s\wedge m+} \leqslant cN^{-4+3s\wedge m}, \\
\big\|(\tilde\phi_\pm)^2\hat\phi_\pm\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} &\leqslant |I|\|\tilde\phi_\pm\|^2_{L^\infty(I,L_x^{2/\epsilon}(\mathbb{R}^2))}\|\hat\phi_\pm\|_{L^\infty(I,L_x^{2/(1-2\epsilon)}(\mathbb{R}^2))} \\
&\leqslant c|I|\|\tilde\phi_\pm\|^{2\epsilon}_{L^\infty(I,L_x^2(\mathbb{R}^2))}\|\tilde\phi_\pm\|^{2-2\epsilon}_{L^\infty(I,H_x^1(\mathbb{R}^2))}\|\hat\phi_\pm\|_{L^\infty(I,H_x^{2\epsilon}(\mathbb{R}^2))} \\
&\leqslant cN^{(1-s\wedge m)(2-2\epsilon)}N^{2\epsilon-m}N^{-5(1-s\wedge m)-\delta} \leqslant cN^{-3+2s\wedge m}, \\
\big\|(\hat\phi_\pm)^3\big\|_{L_t^1(I,L_x^2(\mathbb{R}^2))} &\leqslant |I|^{\frac12}\|\hat\phi_\pm\|^3_{L_t^6(I,L_x^6(\mathbb{R}^2))} \leqslant c|I|^{\frac12}\|\hat\phi_\pm\|^3_{X_\pm^{1/2,(1/2)+}(I)} \\
&\leqslant cN^{-\frac52(1-s\wedge m)-\frac{\delta}{2}}N^{3(1/2-s\wedge m)} \leqslant cN^{-1-\frac12 s\wedge m}.
\end{aligned}$$

From the above estimates, we get the desired result. □

7. The iteration process

In the preceding sections we constructed a solution of problem (1.1) in the time interval $I=[0,|I|]$ with $|I|=N^{-5(1-s\wedge m)-}$. Namely, if we define $\psi:=\hat\psi+\tilde\psi$, $\phi_\pm:=\hat\phi_\pm+\tilde\phi_\pm$ we see that (ψ,ϕ_+,ϕ_-) solves system (3.2) with initial condition $\psi(0)=\psi_0$, $\phi_\pm(0)=\phi_{0\pm}$.

This problem is equivalent to the original system (1.1). The initial data are transformed by $\phi_{0\pm} = \phi_0 \pm iA^{-1}\phi_1$ or, conversely, by $\phi_0 = \frac{1}{2}(\phi_{0+} + \phi_{0-})$, $\phi_1 = -\frac{i}{2}A(\phi_{0+} - \phi_{0-})$. In order to continue the solution of (3.2) we take as new initial data $(\tilde{\psi}(|I|) + w(|I|), \tilde{\phi}_+(|I|) + z_+(|I|), \tilde{\phi}_-(|I|) + z_-(|I|))$ instead of $(\psi_{01}, \phi_{0+1}, \phi_{0-1})$. When we have shown that this problem has a solution $(\tilde{\tilde{\psi}}, \tilde{\tilde{\phi}}_+, \tilde{\tilde{\phi}}_-)$ in the interval $[|I|, 2|I|]$ with equal length $|I|$ we insert this solution into the system (6.1) in place of $(\tilde{\psi}, \tilde{\phi}_+, \tilde{\phi}_-)$ and solve this problem with data $(U(|I|)\psi_{02}, U_-(|I|)\phi_{0+2}, U_+(|I|)\phi_{0-2})$ in $[|I|, 2|I|]$. The solution of the original system (1.1) corresponding to $(\tilde{\tilde{\psi}}, \tilde{\tilde{\phi}}_+, \tilde{\tilde{\phi}}_-)$, denoted by $(\tilde{\tilde{\psi}}, \tilde{\tilde{\phi}})$, then obviously has the following initial data

$$
\begin{aligned}
\tilde{\tilde{\psi}}(|I|) &= \tilde{\psi}(|I|) + w(|I|), \\
\tilde{\tilde{\phi}}(|I|) &= \frac{1}{2}\big(\tilde{\tilde{\phi}}_+(|I|) + \tilde{\tilde{\phi}}_-(|I|)\big) \\
&= \frac{1}{2}\big(\tilde{\phi}_+(|I|) + z_+(|I|) + \tilde{\phi}_-(|I|) + z_-(|I|)\big) \\
&= \tilde{\phi}(|I|) + \frac{1}{2}\big(z_+(|I|) + z_-(|I|)\big) =: \tilde{\phi}(|I|) + z(|I|), \\
\tilde{\tilde{\phi}}_t(|I|) &= -\frac{i}{2}A\big(\tilde{\tilde{\phi}}_+(|I|) - \tilde{\tilde{\phi}}_-(|I|)\big) \\
&= -\frac{i}{2}A\big(\tilde{\phi}_+(|I|) + z_+(|I|) - \tilde{\phi}_-(|I|) - z_-(|I|)\big) \\
&= \tilde{\phi}_t(|I|) - \frac{i}{2}A\big(z_+(|I|) - z_-(|I|)\big) =: \tilde{\phi}_t(|I|) + z_t(|I|).
\end{aligned}
$$

Adding up the solutions, we get a solution of the original problem in $[|I|, 2|I|]$ as before. This defines an iteration process. At each step, the replacement of $(\psi_{02}, \phi_{0+2}, \phi_{0-2})$ by $(U(|I|)\psi_{02}, U_-(|I|)\phi_{0+2}, U_+(|I|)\phi_{0-2})$ is harmless, because the H^s-norms remain unchanged. The bounds on the data are controlled by the energy and the L^2 conservation law. Thus we have to estimate these quantities independently of the iteration step. This is easy for the L^2 conserved quantity, the increment when replacing ψ_{01} by $\tilde{\psi}(|I|) + w(|I|)$ is given by

$$
\begin{aligned}
\big|\|\tilde{\psi}(|I|) + w(|I|)\|_{L^2_x} - \|\psi_{01}\|_{L^2_x}\big| &= \big|\|\tilde{\psi}(|I|) + w(|I|)\|_{L^2_x} - \|\tilde{\psi}(|I|)\|_{L^2_x}\big| \\
&\leqslant \|w(|I|)\|_{L^2_x} \leqslant c_2 N^{-\frac{1}{4}-\frac{3}{4}(s\wedge m)},
\end{aligned}
$$

by Proposition 6.3, where $c_2 = c_2(\bar{c}, M)$.

The number of iteration steps in order to reach the given time T is $\frac{T}{|I|} = TN^{5(1-s\wedge m)+}$. This means that in order to get uniform control over the L^2 norm of $\tilde{\psi}, \tilde{\tilde{\psi}}, \ldots,$ we have to ensure that $c_2 T N^{5(1-s\wedge m)+} N^{-\frac{1}{4}-\frac{3}{4}(s\wedge m)} < M$, where $c_2 = c_2(2\bar{c}, 2M)$ (remark that initially the L^2 norm of $\tilde{\psi}$ was bounded by M). This is fulfilled for N sufficiently large if

$$5(1-s\wedge m) - \frac{1}{4} - \frac{3}{4}(s\wedge m) < 0 \quad \Leftrightarrow \quad s\wedge m > \frac{19}{23},$$

here is the point where the decisive bound on $s\wedge m$ appears.

Concerning the increment of the energy. Because we define $z = \frac{1}{2}(z_+ + z_-)$, $z_t = -\frac{i}{2}A^{\frac{1}{2}}(z_+ - z_-)$ in the above and estimate:

$$\begin{aligned}
&\bigl|E\bigl(\tilde{\psi}(|I|) + w(|I|), \tilde{\phi}(|I|) + z(|I|), \tilde{\phi}_t(|I|) + z_t(|I|)\bigr) - E(\psi_{01}, \phi_{01}, \phi_{11})\bigr| \\
&\quad = \bigl|E\bigl(\tilde{\psi}(|I|) + w(|I|), \tilde{\phi}(|I|) + z(|I|), \tilde{\phi}_t(|I|) + z_t(|I|)\bigr) \\
&\qquad - E\bigl(\tilde{\psi}(|I|), \tilde{\phi}(|I|), \tilde{\phi}_t(|I|)\bigr)\bigr| \\
&\quad \leqslant 2\bigl(\bigl\|\nabla\tilde{\psi}(|I|)\bigr\| + \bigl\|\nabla w(|I|)\bigr\|\bigr)\bigl\|\nabla w(|I|)\bigr\| \\
&\qquad + \bigl(\bigl\|A\tilde{\phi}(|I|)\bigr\| + \bigl\|Az(|I|)\bigr\|\bigr)\bigl\|Az(|I|)\bigr\| + \bigl(\bigl\|\tilde{\phi}_t(|I|)\bigr\| + \bigl\|z_t(|I|)\bigr\|\bigr)\bigl\|z_t(|I|)\bigr\| \\
&\qquad + \int_{\mathbb{R}^2} \bigl|z(|I|)\bigr|\bigl|\tilde{\psi}(|I|) + w(|I|)\bigr|^2\,dx \\
&\qquad + \int_{\mathbb{R}^2} \bigl|\tilde{\phi}(|I|)\bigr|\Bigl|\bigl|\tilde{\psi}(|I|) + w(|I|)\bigr|^2 - \bigl|\tilde{\psi}(|I|)\bigr|^2\Bigr|\,dx \\
&\qquad + \frac{1}{2}\biggl|\int_{\mathbb{R}^2} |\tilde{\psi} + w|^4 - |\tilde{\psi}|^4\,dx\biggr| + \frac{1}{4}\biggl|\int_{\mathbb{R}^2} |\tilde{\phi} + z|^4 - |\tilde{\phi}|^4\,dx\biggr|.
\end{aligned}$$

The first term is bounded by Proposition 6.3 and (4.5) by

$$c\bigl(N^{1-s\wedge m} + N^{-1+\frac{1}{2}(s\wedge m)}\bigr)N^{-1+\frac{1}{2}(s\wedge m)} \leqslant cN^{1-s\wedge m}N^{-1+\frac{1}{2}(s\wedge m)} = cN^{-\frac{1}{2}(s\wedge m)}.$$

The second and third terms are bounded by

$$c\bigl(N^{1-s\wedge m} + N^{-3+2(s\wedge m)}\bigr)N^{-3+2(s\wedge m)} \leqslant cN^{-2+s\wedge m}.$$

By Propositions 6.3 and 6.4 the forth term is estimated as follows:

$$\begin{aligned}
\int_{\mathbb{R}^2} \bigl|z(|I|)\bigr|\bigl|\tilde{\psi}(|I|) + w(|I|)\bigr|^2\,dx &\leqslant c\bigl\|z(|I|)\bigr\|_{L^2}\bigl(\bigl\|\tilde{\psi}(|I|)\bigr\|_{L^4}^2 + \bigl\|w(|I|)\bigr\|_{L^4}^2\bigr) \\
&\leqslant c\bigl\|z(|I|)\bigr\|_{L^2}\bigl(\bigl\|\tilde{\psi}(|I|)\bigr\|_{H^1}^2 + \bigl\|w(|I|)\bigr\|_{H^1}^2\bigr) \\
&\leqslant cN^{-s\wedge m}\bigl(N^{2(1-s\wedge m)} + N^{-2+s\wedge m}\bigr) \\
&\leqslant cN^{2-3(s\wedge m)}.
\end{aligned}$$

The fifth term is estimated by Lemma 5.1, Proposition 6.3 as follows:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} |\tilde{\phi}(|I|)|\big||\tilde{\psi}(|I|)+w(|I|)|^2-|\tilde{\psi}(|I|)|^2\big|\,dx\\
&\quad\leqslant c\|\tilde{\phi}(|I|)\|_{L^4}\|w(|I|)\|_{L^2}\big(\|\tilde{\psi}(|I|)\|_{L^4}+\|w(|I|)\|_{L^4}\big)\\
&\quad\leqslant c\|A\tilde{\phi}(|I|)\|_{L^2}\|w(|I|)\|_{L^2}\big(\|\tilde{\psi}(|I|)\|_{L^2}^{\frac12}\|\nabla\tilde{\psi}(|I|)\|_{L^2}^{\frac12}+\|w(|I|)\|_{L^2}^{\frac12}\|\nabla w(|I|)\|_{L^2}^{\frac12}\big)\\
&\quad\leqslant cN^{1-s\wedge m}N^{-\frac14-\frac34(s\wedge m)}\big(N^{\frac12(1-s\wedge m)}+N^{\frac12(-\frac14-\frac34(s\wedge m))}N^{\frac12(-1+\frac12(s\wedge m))}\big)\\
&\quad\leqslant cN^{1-s\wedge m}N^{-\frac14-\frac34(s\wedge m)}N^{\frac12(1-s\wedge m)}\leqslant cN^{\frac54-\frac94(s\wedge m)}.
\end{aligned}$$

The sixth term is estimated by Lemma 5.1, Proposition 6.3 as follows:

$$\begin{aligned}
&\frac12\bigg|\int\limits_{\mathbb{R}^2}|\tilde{\psi}+w|^4-|\tilde{\psi}|^4\,dx\bigg|\\
&\quad\leqslant 6\big(\|\tilde{\psi}\|_{L_x^4(\mathbb{R}^2)}^3+\|w\|_{L_x^4(\mathbb{R}^2)}^3\big)\|w\|_{L_x^4(\mathbb{R}^2)}\\
&\quad\leqslant 6\big(\|\tilde{\psi}\|_{L_x^2(\mathbb{R}^2)}^{\frac32}\|\nabla\tilde{\psi}\|_{L_x^2(\mathbb{R}^2)}^{\frac32}+\|w\|_{H_x^1(\mathbb{R}^2)}^3\big)\|w\|_{L_x^2(\mathbb{R}^2)}^{\frac12}\|w\|_{H_x^1(\mathbb{R}^2)}^{\frac12}\\
&\quad\leqslant c\big(N^{\frac32(1-s\wedge m)}+N^{3(-1+\frac12 s\wedge m)}\big)N^{\frac12(-\frac14-\frac34(s\wedge m))}N^{\frac12(-1+\frac12 s\wedge m)}\\
&\quad\leqslant cN^{\frac32(1-s\wedge m)}N^{\frac12(-\frac54-\frac14(s\wedge m))}=cN^{\frac78-\frac{13}{8}(s\wedge m)}.
\end{aligned}$$

The seventh term is estimated by Lemma 5.1, Proposition 6.4 as follows:

$$\begin{aligned}
&\frac14\bigg|\int\limits_{\mathbb{R}^2}|\tilde{\phi}+z|^4-|\tilde{\phi}|^4\,dx\bigg|\\
&\quad\leqslant 3\big(\|A\tilde{\phi}\|_{L_x^2(\mathbb{R}^2)}^3+\|z\|_{L_x^2(\mathbb{R}^2)}^{\frac32}\|\nabla z\|_{L_x^2(\mathbb{R}^2)}^{\frac32}\big)\|z\|_{L_x^2(\mathbb{R}^2)}^{\frac12}\|\nabla z\|_{L_x^2(\mathbb{R}^2)}^{\frac12}\\
&\quad\leqslant c\big(N^{3(1-s\wedge m)}+N^{-\frac32 s\wedge m}N^{\frac32(-3+2s\wedge m)}\big)N^{-\frac12 s\wedge m}N^{\frac12(-3+2s\wedge m)}\\
&\quad\leqslant cN^{3(1-s\wedge m)}N^{-\frac12 s\wedge m}N^{\frac12(-3+2s\wedge m)}=cN^{\frac32-\frac52 s\wedge m}.
\end{aligned}$$

Now the forth term behaves better than the first one, because

$$N^{2-3(s\wedge m)}<N^{-\frac12(s\wedge m)}\quad\Leftrightarrow\quad s\wedge m>\frac45.$$

The fifth term is harmless compared to the first term, because

$$N^{\frac54-\frac94(s\wedge m)}\leqslant N^{-\frac12(s\wedge m)}\quad\Leftrightarrow\quad s\wedge m\geqslant\frac57.$$

The sixth term is harmless compared to the first term, because

$$N^{\frac{7}{8}-\frac{13}{8}(s\wedge m)} \leqslant N^{-\frac{1}{2}(s\wedge m)} \quad \Leftrightarrow \quad s\wedge m > \frac{7}{9}.$$

The seventh term is harmless compared to the first term, because

$$N^{\frac{3}{2}-\frac{5}{2}s\wedge m} \leqslant N^{-\frac{1}{2}(s\wedge m)} \quad \Leftrightarrow \quad s\wedge m > \frac{3}{4}.$$

Thus the decisive terms are the first, second and third terms.

Concerning the first term the condition that ensures uniform control of the energy of $(\tilde{\psi},\tilde{\phi})$, $(\tilde{\tilde{\psi}},\tilde{\tilde{\phi}}),\ldots$ is the following

$$c_3 T N^{5(1-s\wedge m)+} N^{-\frac{1}{2}(s\wedge m)} < \bar{c} N^{4(1-s\wedge m)},$$

where $c_3 = c_3(2\bar{c}, 2M)$ (recall that the energy initially is bounded by $\bar{c}N^{4(1-s\wedge m)}$). This is satisfied for N sufficiently large provided $5(1-s\wedge m)+ -\frac{1}{2}(s\wedge m) < 4(1-s\wedge m) \Leftrightarrow s\wedge m > \frac{2}{3}$.

Concerning the second and third term the following condition has to be satisfied:

$$c_3 T N^{5(1-s\wedge m)+} N^{-2+s\wedge m} < \bar{c} N^{4(1-s\wedge m)},$$

this requires $5(1-s\wedge m)+ -2+s\wedge m < 4(1-s\wedge m)$, which is fulfilled.

The uniform control of the energy implies by (4.1) uniform control of the L^2 norm of $(\nabla\tilde{\psi}, A\tilde{\phi}, \tilde{\phi}_t)$, $(\nabla\tilde{\tilde{\psi}}, A\tilde{\tilde{\phi}}, \tilde{\tilde{\phi}}_t),\ldots$.

We have proven:

Theorem 7.1. *Let* $\frac{19}{23} < s, m \leqslant 1$. *Then system* (1.1) *with data*

$$(\psi_0,\phi_0,\phi_1) \in H^s(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2)$$

has a unique global solution. More precisely, for any T *there exists a unique solution*

$$(\psi,\phi,\phi_t) \in X^{s,\frac{1}{2}+}[0,T] \times \widetilde{X}^{m,\frac{1}{2}+}[0,T] \times \widetilde{X}^{m-1,\frac{1}{2}+}[0,T].$$

This solution satisfies

$$(\psi,\phi,\phi_t) \in C^0\big([0,T], H^s(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^{m-1,2}(\mathbb{R}^2)\big).$$

Here $\widetilde{X}^{m,\frac{1}{2}+}[0,T] := X_+^{m,\frac{1}{2}+}[0,T] + X_-^{m,\frac{1}{2}+}[0,T]$.

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